Slide 1

On the Riemann-Poisson manifolds

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Slide 2

Content

1. Definition and examples of Riemann-Poisson manifolds;
2. Some general properties of Riemann-Poisson manifolds;
3. Linear Riemann-Poisson structures;
4. Local structure of flat Riemann-Poisson manifolds;
5. A Riemannian study of Riemann-Poisson structures on dimension 2 or 3.

Slide 3

[1] **M. Boucetta,** Compatibilité des structures pseudo-riemanniennes et des structures de Poisson, C. R. Acad. Sci. Paris, **t. 333** (2001) 763-768.

[2] **M. Boucetta,** Riemann-Poisson manifolds and Kähler-Riemann foliations, C. R. Acad. Sci. Paris, **t. 336** (2003) 423-428.

[3] **M. Boucetta,** Poisson manifolds with compatible pseudo-metric and pseudo-Riemannian Lie algebras, Differential Geometry and its Applications, **Vol. 20** (2004) 279-291.

[4] **M. Boucetta,** On the Riemann-Lie Algebras and Riemann-Poisson Lie Groups, Journal of Lie Theory **Vol. 15** (2005) 183-195.

[5] **M. Boucetta,** A Riemannian study of Poisson structures compatible with a Riemannian metric, in preparation.

Slide 4

Notations

For a Riemannian manifold P of dimension n, we will denote:

- g the metric when it measures the length of tangent vectors;

- $<,>$ the metric when it measures the length of 1-forms;

- ∇ the Levi-Civita covariant connection associated to the metric;

- Ω the Riemannian volume form when P is orientable;

- $\# : T^*P \longrightarrow TP$ the musical isomorphism associated to the metric.

Slide 5

Definition and examples of Riemann-Poisson manifolds

Let P be a Riemannian manifold and π a Poisson tensor. The Poisson tensor determines a Lie algebroid structure on T^*P defined by the vector bundle homomorphism $\pi_\# : T^*P \longrightarrow TP$ given by

$$\beta\left(\pi_\#(\alpha)\right) = \pi(\alpha, \beta),$$

and the Koszul bracket $[\ ,\ ]_\pi$ of 1-forms given by

$$[\alpha, \beta]_\pi = L_{\pi_\#(\alpha)}\beta - L_{\pi_\#(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

$(T^*P, \pi_\#, [\ ,\ ]_\pi, <,>)$ is an euclidian Lie algebroid.

Slide 6

There exists an unique contravariant connection D associated to this euclidian algebroid such that:

$i)$ the metric is parallel with respect to D i.e.

$$\pi_\#(\alpha). < \beta, \gamma > = < D_\alpha\beta, \gamma > + < \beta, D_\alpha\gamma >,$$

$ii)$ D is torsion-free i.e.

$$D_\alpha\beta - D_\beta\alpha = D_{[\alpha,\beta]_\pi}.$$

The connection D will be called **the Levi-Civita contravariant connection** associated with $(<,>, \pi)$.

Slide 7

It is given by

$$\begin{aligned} 2 < D_\alpha\beta, \gamma > &= \pi_\#(\alpha). < \beta, \gamma > + \pi_\#(\beta). < \alpha, \gamma > - \pi_\#(\gamma). < \alpha, \beta > \\ &+ < [\gamma, \alpha]_\pi, \beta > + < [\gamma, \beta]_\pi, \alpha > + < [\alpha, \beta]_\pi, \gamma > . \end{aligned}$$

The symplectic case

When π is invertible, D is given by

$$D_\alpha\beta = \pi_\#^{-1}\left(\nabla^\pi_{\pi_\#(\alpha)}\pi_\#(\beta)\right)$$

where ∇^π is the Levi-Civita connection associated to the metric

$$g_\pi(u,v) =< \pi_\#^{-1}(u), \pi_\#^{-1}(v) > .$$

Slide 8

The triple $(P, <,>, \pi)$ is called a **Riemann-Poisson manifold** if

$$D\pi = 0.$$

A Riemann-Poisson manifold $(P, <,>, \pi)$ will be called **flat** if the curvature tensor of D given by

$$R(\alpha,\beta)\gamma = D_{[\alpha,\beta]_\pi}\gamma - (D_\alpha D_\beta - D_\beta D_\alpha)\gamma$$

vanishes identically.

Slide 9

In the symplectic case, we have

$$D\pi = 0 \Leftrightarrow \nabla\pi = 0,$$

so Riemann-Poisson manifolds are a generalization of Kähler manifolds.

In general case, we have

$$\nabla\pi = 0 \qquad \Rightarrow \qquad D\pi = 0.$$

Slide 10

Example

Let P be a Riemannian manifold. Consider

$$\pi = \sum_{i<j} X_i \wedge X_j.$$

where $(X_1, \ldots, X_p)$ is a family of commuting Killing vector fields.

We have

$$D_\alpha\beta = \sum_{i<j} \alpha(X_i) L_{X_j}\beta - \alpha(X_j) L_{X_i}\beta$$

and

$$D\pi = 0.$$

Moreover, $(P, <, >, \pi)$ is a flat Riemann-Poisson manifold.

Slide 11

Let us specify a particular case of this situation.

Theorem 0.1 *Let $<,>$ be a bi-invariant metric on a Lie group G and $r \in \wedge^2\mathcal{G}$ such that $[r,r]=0$. Then the following assertions are equivalent:*

(i) $(G, r^l, <,>)$ *is a Riemann-Poisson manifold.*

(ii) $(G, r^r, <,>)$ *is a Riemann-Poisson manifold.*

(iii) $(G, r^l - r^r, <,>)$ *is a Riemann-Poisson manifold.*

(iv) Imr *is an abelian subalgebra.*

Moreover, all this three structures are flat.

Slide 12

Proposition 0.1 Let $<,>$ be a bi-invariant metric on a Lie group G and $r \in \wedge^2\mathcal{G}$. Then the following assertions are equivalent:

(i) $\nabla r^l = 0$.

(ii) r is $ad_{\mathcal{G}}$-invariant.

In this case, Imr is an abelian ideal.

Slide 13

Some general properties of Riemann-Poisson manifolds

Slide 14

Proposition 0.2 *Let P be a Riemann-Poisson manifold. Then the symplectic leaves are Kähler. Moreover, if the Riemann-Poisson is flat then each symplectic leaf is affine.*

Slide 15

Let $(P, \pi, <,>)$ be a Riemann-Poisson manifold. We have:

$\diamond$ $p \in P$ is a regular point and $\alpha \in T_p^*P$ with $\pi_\#(\alpha) = 0$ then $D_\alpha = 0$.

$\diamond$ f is a Casimir funtion then

$$Ddf = 0 \qquad \text{and} \qquad [\#(df), \pi] = 0.$$

Slide 16

$\diamond$ Let U be un open set where the rank of π is constant. Put, in restriction to U,

$$T^*P = Ker\pi_\# \oplus Ker\pi_\#^\perp.$$

Then:

1. $\alpha, \beta \in \Gamma(Ker\pi^\perp)$ then $D_\alpha\beta \in \Gamma(Ker\pi^\perp)$,
2. $\Gamma(Ker\pi^\perp)$ is subalgebra of $(\Omega^1(U), [\ ,\ ]_\pi)$.

Slide 17

Theorem 0.2 *For any regular Riemann-Poisson manifold, the associated Lie algebroid is integrable.*

Theorem 0.3 *Let P be a Riemann-Poisson manifold. Denote by μ the Riemannian density. Then, for any function $f \in C^\infty(P, \mathbb{R})$,*

$$L_{X_f}\mu = 0.$$

This means that the Poisson structure is unimodular with respect to the Riemannian density.

Slide 18

Theorem 0.4 *Let P be a Riemann-Poisson manifold such that the Poisson tensor is regular. Denote by $\mathcal{F}$ the symplectic foliation and by ω the leafwise symplectic form. Then there exists a Riemannian metric g_π on P such that*

(i) g_π is a bundle-like metric and then $\mathcal{F}$ is a Riemannian foliation;

(ii) the restrictions of ω and g_π to any symplectic leaf define a Kähler structure;

(iii) for any local perpendicular foliation-preserving vector field X and any couple (U, V) of local vectors fields tangent to $\mathcal{F}$,

$$L_X\omega(U, V) = 0.$$

Slide 19

Linear Riemann-Poisson structures

Let $\mathcal{G}$ be a Lie algebra. To any bilinear symmetric non-degenerate 2-form $<,>$ on $\mathcal{G}$, we will associate a product on $\mathcal{G}$ denoted by $A : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ defined by

$$2 < A_u v, w > = < [u, v], w > + < [w, u], v > + < [w, v], u > .$$

Definition 0.1 *A Lie algebra $(\mathcal{G}, <,>)$ endowed with a scalar product $<,>$ will be called Riemann-Lie algebra if*

$$[A_u v, w] + [v, A_u w] = [u, [v, w]].$$

Slide 20

Proposition 0.3 *Let P be a Riemann-Poisson manifold and $p \in P$. Then the isotropy Lie algebra at p is a Riemann-Lie algebra.*

Proposition 0.4 *Let $\mathcal{G}$ be a Lie algebra. The following assertions are equivalent:*

(i) The dual $\mathcal{G}^*$ carries a Riemannian metric compatible with the linear canonical Poisson structure on $\mathcal{G}^*$.

(ii) There exists on $\mathcal{G}$ a scalar product $<,>$ such that $(\mathcal{G}, <,>)$ is a Riemann-Lie algebra.

Slide 21

Let G be a Lie group and $\mathcal{G}$ its Lie algebra. Consider a scalar product $<,>$ on $\mathcal{G}$. We will denote by:

- $<,>^*$ the Riemannian metric on $\mathcal{G}^*$ associated to $<,>$;

- $<,>^l$ the left-invariant Riemannian metric on G associated to $<,>$ and ∇ its Levi-Civita connection;

- θ the right-invariant Maurer-Cartan form on G;

- $S_{<,>} = \{u \in \mathcal{G}; ad_u + ad_u^t = 0\}$.

Slide 22

Theorem 0.5 *Let G be a Lie group, $(\mathcal{G}, [\ ,\ ])$ its Lie algebra and $<,>$ a scalar product on $\mathcal{G}$. Then, the following assertions are equivalent:*

1) $(\mathcal{G}, [\ ,\ ], <,>)$ is a Riemann-Lie algebra.

2) $(\mathcal{G}^*, \pi^l, <,>^*)$ is a Riemann-Poisson manifold (π^l is the canonical Poisson tensor on $\mathcal{G}^*$).

3) The 2-form $d\theta \in \Omega^2(G, \mathcal{G})$ is parallel with respect the Levi-Civita connection ∇ i.e. $\nabla d\theta = 0$.

4) $(G, <,>^l)$ is a flat Riemannian manifold.

5) The orthogonal subalgebra $S_{<,>}$ of $(\mathcal{G}, [\ ,\ ], <,>)$ is abelian and $\mathcal{G}$ split as an orthogonal direct sum $S_{<,>} \oplus \mathcal{U}$ where $\mathcal{U}$ is a commutative ideal.

Slide 23

Consider the Heisenberg Lie algebra $H_3 = \mathbb{R}^3$ where the bracket is given by

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.$$

Define the Lorentzian bilinear 2-form on H_3 by

$$< u, v >=< Su, v >_0$$

where

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and $<,>_0$ the usual scalar product on $\mathbb{R}^3$.

$(H_3, <,>)$ is a Lorentz-Lie algebra.

Slide 24

The local structure of flat

Riemann-Poisson manifolds

Theorem 0.6 *Let $(P, <,>, \pi)$ be a flat Riemann-Poisson manifold. Let p be a regular point where the rank of π is locally constant equal to $2r$. Then there exists on a neighborhood of p a family $(X_1, \ldots, X_r, Y_1, \ldots, Y_r)$ of commuting Killing vectors fields such that*

$$\pi = \sum_{j=1}^{r} X_j \wedge Y_j.$$

Slide 25

A Riemannian study of Riemann-Poisson manifolds on dimension 2 or 3

Slide 26

Let P be an oriented Riemannian manifold and Ω the Riemannian volume. Let $\omega \in \Omega^2(P)$ and put $\pi = \#(\omega)$.

π is a Poisson tensor iff

$$\delta(\omega \wedge \omega) = 2\omega \wedge \delta(\omega).$$

π is unimodular with respect to Ω iff

$$di_\pi \Omega = d * \omega = 0.$$

Thus π is a Poisson tensor unimodular with respect to Ω iff

$$\delta(\omega) = 0 \qquad \text{and} \qquad \delta(\omega \wedge \omega) = 0.$$

Slide 27

For $1 \leq p \leq \left[\frac{dimP}{2}\right]$, we define a multi-vector Q_p by

$$i_{Q_p}\Omega = \wedge^p\omega.$$

Let us recall the following formula

$$i_{[\pi,Q_p]}\Omega = -\sigma(i_{Q_p}\Omega)$$

where σ is the Koszul-Brylinski given by $\sigma = i_\pi \circ d - d \circ i_\pi$. We have

$$D\pi = 0 \quad \Leftrightarrow \quad D\omega = 0 \quad \Leftrightarrow \quad DQ_1 = 0.$$

$D\omega = 0$ implies $D(\wedge^p\omega) = 0$ which is equivalent to $DQ_p = 0$. Since D is torsion-free, $DQ_p = 0$ implies $[\pi, Q_p] = 0$ which is equivalent to

$$\sigma(\wedge^p\omega) = 0.$$

Slide 28

Proposition 0.5 *Let $(P, <,>, \pi)$ be a Riemann-Poisson oriented n-manifold. Consider the differential 2-form ω obtained form π when one identifies, by the metric, the tangent and the cotangent bundles. Then ω satisfies the following necessary conditions:*

$$\delta(\omega) = 0, \quad \delta(\omega \wedge \omega) = 0 \quad and$$

$$\sigma(\wedge^p\omega) = 0 \quad for \quad p = 1 \dots [\frac{n}{2}], \tag{1}$$

where δ is the divergence operator associated to the metric and σ is the Koszul-Brylinski operator given by $\sigma = i_\pi \circ d - d \circ i_\pi$.

Slide 29

In dimension 2, the conditions (1) reduce to the condition $\delta(\omega) = 0$ which implies that $\omega = c\Omega$ where c is a real constant. This gives all Poisson structures compatible with a Riemannian metric on an oriented surface.

Slide 30

In dimension 3, if we replace ω by $\alpha = *\omega$, where $*$ is the Hodge operator, we can writ the conditions (1) in a different manner, show that this conditions are in fact sufficient and get the following result.

Theorem 0.7 *Let $(P, <,>)$ be an oriented Riemannian 3-manifold and let π be a bivector field on P. Consider the differential 1-form α given by $\alpha = i_\pi\Omega$. Then $(P, <,>, \pi)$ is a Riemann-Poisson manifold iff*

$$d\alpha = 0 \quad \textit{and} \quad d < \alpha, \alpha > + \delta(\alpha)\alpha = 0 \quad \Leftrightarrow \quad ([\#(\alpha), \pi] = 0).$$

Slide 31

Corollary 0.1 *Let $(P,<,>)$ be an oriented Riemannian 3-manifold such that $H^1_{dR}(P)=0$ and let π be a bivector field on P. Then $(P,<,>,\pi)$ is a Riemann-Poisson manifold if and only if there exists $f\in C^\infty(P)$ such that $i_\pi\Omega=df$ and*

$$d<df,df>+\Delta(f)df=0 \quad \Leftrightarrow \quad ([\#(df),\pi]=0)$$

where Δ is the Beltrami-Hodge Laplacian acting on functions.

Slide 32

Examples. According to Corollary 0.1, a Poisson tensor π on $\mathbb{R}^3$ is compatible with the Euclidian metric if and only if

$$\pi=\frac{\partial f}{\partial z}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\frac{\partial f}{\partial x}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$$

where $f\in C^\infty(\mathbb{R}^3)$ and verifies

$$d<df,df>+\Delta(f)df=0. \qquad (E)$$

Slide 33

The polynomial functions of degree 2 solutions of (E) are

$$\begin{aligned} f(x,y,z) &= (a+c)x^2 + (a+b)y^2 + (b+c)z^2 \\ &- 2\sqrt{bc}xy + 2\sqrt{ab}xz + 2\sqrt{ac}yz. \end{aligned}$$

This gives all linear Poisson structures on $\mathbb{R}^3$ compatible with the Euclidian metric.

Slide 34

Let $\pi_{so(3)} = z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ be the linear Poisson structure on $\mathbb{R}^3$ corresponding to the Lie algebra $so(3)$. In Theorem 0.3, we have shown that there isn't any Riemannian metric on $\mathbb{R}^3$ compatible with $\pi_{so(3)}$. However, we have the following proposition.

Proposition 0.6 *The function*

$$f(x,y,z) = (x^2 + y^2 + z^2)^{\frac{3}{2}}$$

is a solution of (E) *and then*

$$(x^2 + y^2 + z^2)^{\frac{1}{2}} \pi_{so(3)}$$

is compatible with the canonical metric of $\mathbb{R}^3$.