The notion of moment map

for families of symplectomorphisms

Moment maps. Review

(M, ω) $\quad f \in \text{Symplecto}(M)$

f is Hamiltonian $\Rightarrow$ $\exists$ an isotopy

$f_t \in \text{Symplecto}(M)$ $\quad f_0 = id$, $f_1 = f$

such that the vector field

$$v_t = f_t^{-1} \frac{df_t}{dt}$$

is Hamiltonian

Thm: $\mathrm{Ham}(M)$ is a normal subgroup of $\mathrm{Symplecto}(M)$

The classical notion of moment map:

G a Lie group, $\mathfrak{g} = \mathrm{Lie}\, G$

$\tau : G \longrightarrow \mathrm{Ham}(M)$

$d\tau : \mathfrak{g} \longrightarrow$ Ham. vector fields

③

$$d\tau(v) = v_M \iff \iota(v_M)\,\omega = d\phi^v$$

<u>Remark</u>: ϕ^v is defined up to an additive constant. One needs some intrinsic way of fixing this constant.

The moment map

$$\Phi : M \to \mathfrak{g}^*$$

$$\langle \Phi, v \rangle = \phi^v$$

What happens if we replace G by a manifold S and φ by a map

$$\varphi : S \to \mathrm{Ham}(M)$$

<u>Claim</u>

1) One can still associate a moment map to φ

2) This map is interesting

The symplectic "category":

The objects in this category:

symplectic manifolds

The morphisms: canonical relations

i.e Given objects M_1, M_2, a morphism

$$\Gamma : M_1 \Rightarrow M_2$$

is a Lagrangian submanifold of $M_1^- \times M_2$

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Composition of morphisms:

$$\Gamma_1 : M_1 \Rightarrow M_2$$

and

$$\Gamma_2 : M_2 \Rightarrow M_3$$

compose to give

$$\Gamma_2 \circ \Gamma_1 : M_1 \Rightarrow M_3$$

where $(m_1, m_3) \in \Gamma_2 \circ \Gamma_1 \iff \exists\, m_2 \in M_2$ s.t.

for $i = 1, 2$ $\quad (m_i, m_{i+1}) \in \Gamma_i$

<u>Theorem</u>: Modulo clean intersection hypotheses, $\Gamma_2 \circ \Gamma_1$ is a canonical relation

Remark Γ_1 and Γ_2 might <u>not</u> satisfy these hypotheses (in which case we can't compose them.) Thus "symp" is not a category but just a "category" in quotation marks.

Comments

1. "symp" is a <u>pointed</u> category:

it contains a "point object", pt:

the unique connected zero dimensional

symplectic manifold, pt.

2. Let M be an object in the category "symp". The "points" of M are its "categorical points", i.e a "point" of M is a morphism

$$\Gamma : pt \Rightarrow M \quad ,$$

in other words, a <u>Lagrangian submanifold</u> of $pt \times M = M$.

3. Thus, modulo clean intersection hypotheses, a morphism, $\Gamma : M_1 \Rightarrow M_2$, maps "points" of M_1 to "points" of M_2.

4. From $\Gamma : M_1 \Rightarrow M_2$ one gets a transpose morphism

$$\Gamma^t : M_2 \Rightarrow M_1$$

where $(m_2, m_1) \in \Gamma^t$ iff $(m_1, m_2) \in \Gamma$

5. For more about "symp" see

Weinstein, Bull. AMS (1981)

6. There is a linear version of "symp" in which objects are symplectic vector spaces and morphisms are <u>linear</u> canonical relations. (This is, in fact, a category: <u>no</u> quotation marks.) See

Gu.- Sternberg, AMJ (1979)

Back to moment maps:

Via the trivialization $T^*G = \mathfrak{g}^* \times G$

$\Phi : M \to \mathfrak{g}^*$ extends to a map

$$\underline{\Phi} : M \times G \to T^*G$$

Let $\Gamma_{\Phi} \subseteq M \times M^- \times T^*G$ be

the set of all triples (m_1, m_2, μ) with

$$m_2 = \tau_g m_1 \quad \text{and} \quad \mu = \underline{\Phi}(m_1, g)$$

Theorem (Weinstein) Γ_{Φ} is a canonical relation, i.e. a morphism

$$\Gamma_{\Phi} : M^{-} \times M \Rightarrow T^{*}G$$

Recall: From Γ_{Φ} one gets a map of "points" of $M^{-} \times M$ to "points" of $T^{*}G$ and from Γ_{Φ}^{t} a map of "points" of $T^{*}G$ to "points" of $M^{-} \times M$.

Example 1 : Δ_M in $M^- \times M$ corresponds to a Lagrangian manifold, Λ_Φ, in T^*G the <u>character Lagrangian</u>. For M compact and G a torus, Λ_Φ is the union

$$\bigcup (G_i \times \Delta_i) \subseteq G \times \mathfrak{g}^* = T^*G$$

where the Δ_i's are the faces of the moment polytope and the G_i's their stabilizers

Example 2: $G \times \{0\}$ in T^*G corresponds to a Lagrangian manifold, Γ_G, in $M^- \times M$, i.e a morphism

$$\Gamma_G : M \Rightarrow M$$

Here $\Gamma_G = \Gamma_{red}^t \circ \Gamma_{red}$

where $\Gamma_{red} : M \Rightarrow M_{red}$ is the canonical relation:

$$\{ (m_1, m_2) \, ; \; m_1 \in \Phi^{-1}(0), \; m_2 = \pi_{red}(m_1) \}$$

Generalizations

$$\mathcal{P} : \mathcal{S} \longrightarrow \mathrm{Ham}(M)$$

For each $s \in \mathcal{S}$ and $w \in T_s\mathcal{S}$ let $\gamma(t)$

be a curve with $\gamma(0) = s$ and $\frac{d}{dt}\gamma(0) = w$

Let $f_t = \mathcal{P}(s(t))$ and let

$$v_{s,w} = \left(f^{-1} \circ \frac{df_t}{dt} \right)_{t=0}$$

Then :

(17)

$$i(v_{s,w})\,\omega = d\alpha_{s,w}$$

and we define

$$\Phi : M \times S \longrightarrow T^*S$$

by setting

$$\Phi(m,s) = \xi \in T_s^*$$

where $\langle \xi, w \rangle = \alpha_{s,w}(m)$

<u>Remark</u>: As before $\alpha_{s,w}$ is only defined up to an additive constant and one needs some intrinsic way of fixing this constant.

Now let $\Gamma_{\Phi} \subseteq M \times M^{-} \times T^{*}S$ be the set of all triples (m_1, m_2, μ) with

$m_2 = \tau_s(m_1)$ and $\mu = \Phi(m_1, s)$

In the group case Γ_Φ is a canonical relation. Is this true here?

No. However, modulo the vanishing of a homotopy obstruction Φ can be modified to make this true.

Let $\tilde{\omega}$ be the symplectic form

on $M \times M^- \times T^*S$ and let

$$K : M \times S \to M \times M^- \times T^*S$$

be the imbedding,

$$(m, s) \to (m, \tau_s m, \Phi(m, s))$$

mapping $M \times S$ onto Γ_Φ

Let π_S be the projection : $M \times S \to S$.

<u>Theorem</u> There exist a two-form, $\mu \in \Omega^2(S)$ such that

$$K^*\tilde{\omega} = \pi^*\mu$$

Remark $[\mu] \in H^2(S, \mathbb{R})$ is a homotopy invariant of $\gamma : S \to Ham(M)$

Suppose $[\omega] = 0$ i.e $\omega = d\alpha$.

Modify Φ by setting

$$\Phi_{new}(p,s) = \Phi_{old}(p,s) + \alpha_s$$

Then for this modified Φ, Γ_Φ

is a Lagrangian submanifold of

$M \times M^- \times T^*S$

Thus Γ_Φ is a morphism:

$$\Gamma_\Phi : M^- \times M \Rightarrow T^*S$$

and maps "points" of $M^- \times M$ to "points" of T^*S. Moreover, Γ_Φ^t maps "points" of T^*S to "points" of $M^- \times M$

What is the analogue in this setting of the character Lagrangian?

i.e the Lagrangian manifold, Λ_Φ in T^*S corresponding to Δ_M?

Suppose f is in $\mathrm{Ham}(M)$:

$f : M \to M$

$\tilde{M}$ = the mapping torus of f

<u>Theorem</u> Modulo hypotheses on M, f

$\tilde{M}$ is a contact manifold and f

induces on $\tilde{M}$ a contact flow.

Remarks

1. This is a symplectic version of the standard "mapping torus" constuction of Smale.

2. Just as in the standard mapping torus construction a fixed point p of f correspond to a period trajectory of the flow on $\tilde{M}$.

3. We'll denote by T_p the period of this trajectory

Now suppose that the map,

$$M \times S \longrightarrow M \times M, \quad (m,s) \to (m, \gamma_s m)$$

is proper and transverse to Δ_M

By Thom transversality

$$\text{graph } \tau_s \pitchfork \Delta_M$$

for $s \in S_0$: an open dense set.

Let

$$p_{i,s} \;, \quad i = 1, 2, \ldots .$$

be the fixed points of τ_s and let

$$\Psi_i(s) = T_{p_{i},s}$$

N.B. $\psi_u \in C^\infty(S_0)$

<u>Theorem</u> $\Lambda_\Phi \cap T^* S_0$ is the union of the Lagrangian manifolds

$$\Lambda_{\psi_u} = \{ (s, \xi) \; ; \; s \in S_0 , \; \xi = (d\psi_i)_s \}$$

Concluding remarks

1. The pre-quantum "category".

This category is identical with the symplectic category except that each morphism is equipped with a symbol: a $\frac{1}{2}$ density times a Maslov factor

The constructions above work in this category: In particular given a Lagrangian manifold in T^*S one can by Γ_Φ^t associate to it a morphism, $M \Rightarrow M$ and take the "symbolic trace" of this morphism.

2. The set $Can(M)$ of canonical relations, $\Gamma : M \rightrightarrows M$, is a "group" (under composition of canonical relations.) A lot of the results above are true with $Ham(M)$ replaced by $Can(M)$.

3. Let G be a non-compact Lie group and $\mathcal{V}: G \to \mathrm{Ham}(M)$ a Hamiltonian action in the usual sense. Sitting in G are the sets of conjugacy classes

(*) S_{elliptic}, $S_{\text{hyperbolic}}$, $S_{\text{unipotent}}$ etc.

Recent results of Weinstein suggest that M has various "different kinds" of moment geometry

i.e. there is a moment geometry attached to the map

$$\Psi : S \longrightarrow \mathrm{Ham}(M)$$

for each of the sets $(*)$.

3. Let G be a non-compact Lie group and $\varphi: G \to \mathrm{Ham}(M)$ a Hamiltonian action in the usual sense. Sitting in G are the sets of conjugacy classes

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