POISSON STRUCTURES AND BIHERMITIAN METRICS

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- $X + \xi \in C^{\infty}(T \oplus T^*)$

- *COURANT bracket*

  $$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)$$

1. GENERAL PHILOSOPHY

2. GERBES

3. GENERALIZED COMPLEX STRUCTURES

4. GENERALIZED KÄHLER STRUCTURES

5. OPEN QUESTIONS

GENERAL PHILOSOPHY

BASIC SCENARIO

- manifold M^n

- replace T by $T \oplus T^*$

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BASIC SCENARIO

- manifold M^n

- replace T by $T \oplus T^*$

- inner product of signature (n, n)

$$(X + \xi, X + \xi) = -i_X \xi$$

- skew adjoint transformations:

$$\mathrm{End}\, T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

- .... *in particular* $B \in \Lambda^2 T^*$

TRANSFORMATIONS

- exponentiate B:

  $X + \xi \mapsto X + \xi + i_X B$

- *$B \in \Omega^2$ ... the B-field*

- natural group $\mathrm{Diff}(M) \ltimes \Omega^2(M)$

SPINORS

- Take $S = \Lambda^* T^*$

- $S = S^{ev} \oplus S^{od}$

- Define Clifford multiplication by

$$\begin{aligned} (X+\xi)\cdot\varphi &= i_X\varphi + \xi\wedge\varphi \\ (X+\xi)^2\cdot\varphi &= i_X\xi\varphi = -(X+\xi, X+\xi)\varphi \end{aligned}$$

- $\exp B(\varphi) = (1 + B + \frac{1}{2}B\wedge B + \ldots)\wedge\varphi$

DERIVATIVES

- Lie bracket:

  $$2i_{[X,Y]}\alpha = d([i_X, i_Y]\alpha) + 2i_X d(i_Y\alpha) - 2i_Y d(i_X\alpha) + [i_X, i_Y]d\alpha$$

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- $A = X + \xi$, $B = Y + \eta$ use Clifford multiplication $A \cdot \alpha$ to define a bracket $[A, B]$:

  $$2[A, B] \cdot \alpha = d((A \cdot B - B \cdot A) \cdot \alpha) + 2A \cdot d(B \cdot \alpha) -$$

  $$-2B \cdot d(A \cdot \alpha) + (A \cdot B - B \cdot A) \cdot d\alpha$$

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- *COURANT bracket*

  $$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)$$

Apply a 2-form B...

- $d \mapsto e^{-B} d e^{B} = d + dB$

- $[X+\xi, Y+\eta] \mapsto [X+\xi, Y+\eta] - 2 i_X i_Y dB$

- $\mathrm{Diff}(M) \ltimes \Omega^2_{closed}(M)$ preserves inner product, exterior derivative and Courant bracket.

GENERALIZED GEOMETRIC STRUCTURES

- $SO(n,n)$ compatibility
- integrability $\sim d$ or Courant bracket
- transform by Diff$(M) \ltimes \Omega^2_{closed}(M)$

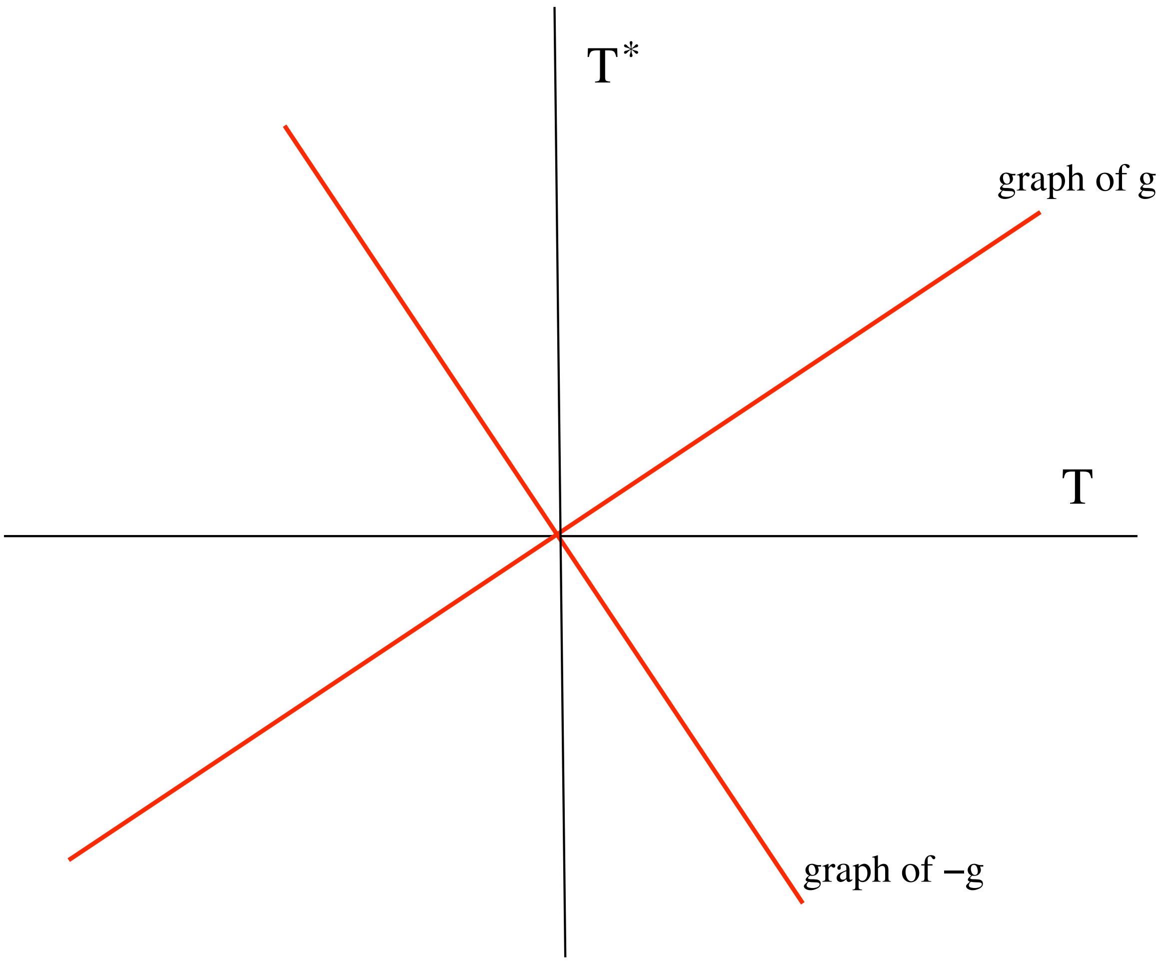
RIEMANNIAN METRIC

- Riemannian metric g_{ij}

- $X \mapsto g(X, -) : g : T \to T^*$

- *graph* of $g = V \subset T \oplus T^*$

- $T \oplus T^* = V \oplus V^\perp$


T*
graph of g
T
graph of −g

GENERALIZED RIEMANNIAN METRIC

- $V \subset T \oplus T^*$ positive definite rank n subbundle

- $=$ graph of $g + B : T \to T^*$

- $g + B \in T^* \otimes T^*$: g symmetric, B skew

GERBES

GERBES: ČECH 2-COCYCLES

- $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$
- $(g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = \ldots)$
- $\delta g = g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

This *defines* a gerbe.

CONNECTIONS ON GERBES

Connective structure:

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$$

Curving:

$$B_\beta - B_\alpha = dA_{\alpha\beta}$$

$\Rightarrow dB_\beta = dB_\alpha = H|_{U_\alpha}$ global three-form H

J.-L. Brylinski, *Characteristic classes and geometric quantization*, Progr. in Mathematics **107**, Birkhäuser, Boston (1993)

TWISTING $T \oplus T^*$

$$dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = d[g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}] = 0$$

- identify $T \oplus T^*$ on U_α with $T \oplus T^*$ on U_β by

  $X + \xi \mapsto X + \xi + i_X dA_{\alpha\beta}$

- defines a vector bundle E

  $0 \to T^* \to E \to T \to 0$

- with ... an inner product and a Courant bracket.

TWISTED COHOMOLOGY

- identify $\Lambda^* T^*$ on U_α with $\Lambda^* T^*$ on U_β by

  $\varphi \mapsto e^{dA_{\alpha\beta}}\varphi$

- defines a vector bundle $S =$ spinor bundle for E

- $d : C^\infty(S) \to C^\infty(S)$ well-defined

- $\ker d / \operatorname{im} d =$ twisted cohomology.

... WITH A CURVING

- $\varphi_\alpha = e^{dA_{\alpha\beta}}\varphi_\beta$
- $d\varphi_\alpha = 0$
-
-
-

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- $e^{B_\alpha}\varphi_\alpha = e^{B_\beta}\varphi_\beta = \psi$

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... WITH A CURVING

- $\varphi_\alpha = e^{dA_{\alpha\beta}} \varphi_\beta$
- $d\varphi_\alpha = 0$
- Curving: $B_\beta - B_\alpha = dA_{\alpha\beta}$
- $e^{B_\alpha} \varphi_\alpha = e^{B_\beta} \varphi_\beta = \psi$
- $d\psi + H \wedge \psi = 0$

Definition: A **generalized metric** is a subbundle $V \subset E$ such that $\mathrm{rk}\, V = \dim M$ and the inner product is positive definite on V.

- $V \cap T^* = 0 \Rightarrow$ splitting of the sequence

  $0 \to T^* \to E \to T \to 0$

- $V^{\perp} \subset E$ another splitting

- difference $\in \mathrm{Hom}(T, T^*) = T^* \otimes T^* =$ Riemannian metric

SPLITTINGS IN LOCAL TERMS

- splitting: $C_\alpha \in C^\infty(U_\alpha, T^* \otimes T^*)$: $C_\beta - C_\alpha = dA_{\alpha\beta}$
- $Sym(C_\alpha) = Sym(C_\beta) =$ metric
- $Alt(C_\alpha) = B_\alpha =$ curving of the gerbe
- $H = dB_\alpha$ closed 3-form

- *two* splittings V and $V^{\perp}$ of $0 \to T^* \to E \to T \to 0$
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- Courant bracket $[X^{+}, Y^{-}]$, Lie bracket $[X, Y]$
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- X vector field, lift to $X^+ \in C^\infty(V)$ and $X^- \in C^\infty(V^\perp)$ in E
- Courant bracket $[X^+, Y^-]$, Lie bracket $[X, Y]$
- $[X^+, Y^-] - [X, Y]^+$ is a one-form

- $[X^+, Y^-] - [X, Y]^+ = 2g(\nabla^+_X Y)$
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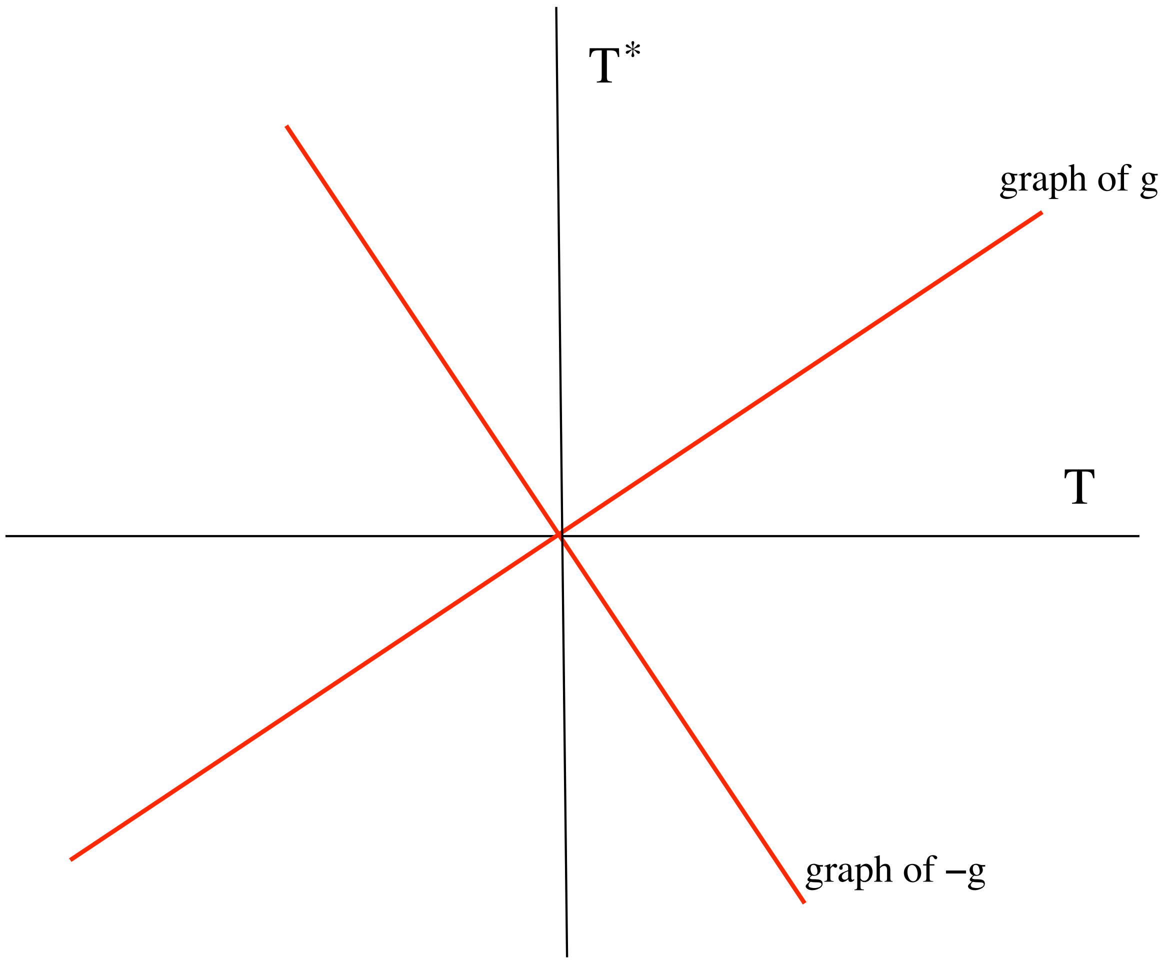
- $[X^-, Y^+] - [X, Y]^- = -2g(\nabla^-_X Y)$ has skew torsion $-H$



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EXAMPLE: the Levi-Civita connection

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k\right] - \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]^+ =$$

$$= \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) dx_k$$

GENERALIZED COMPLEX STRUCTURES

A *generalized complex structure* is:

- $J : T \oplus T^* \to T \oplus T^*, J^2 = -1$

- $(JA, B) + (A, JB) = 0$

- if $JA = iA, JB = iB$ then $J[A, B] = i[A, B]$ (*Courant bracket*)

- $U(m, m) \subset SO(2m, 2m)$ structure on $T \oplus T^*$

M. Gualtieri: *Generalized complex geometry* math.DG/0401221

EXAMPLES

- complex manifold $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$J = i : [\ldots \partial/\partial z_i \ldots, \ldots d\bar{z}_i \ldots]$$

- symplectic manifold $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$$J = i : [\ldots, \partial/\partial x_j + i \sum \omega_{jk} dx_k, \ldots]$$

A POISSON STRUCTURE

- $\{f, g\} = -(Jdf, dg)$ is a Poisson structure
- $Jdf = X_f + \xi$
- integrability: $[Jdf, Jdg] - J[Jdf, dg] - J[df, Jdg] - [df, dg] = 0$
- $\Rightarrow X_{\{f,g\}} = [X_f, X_g]$

EXAMPLE: HOLOMORPHIC POISSON MANIFOLDS

$$\sigma = \sum \sigma^{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

$$[\sigma, \sigma] = 0$$

$$J = i : \left[\ldots, \frac{\partial}{\partial z_j}, \ldots, d\bar{z}_k + \sum_{\ell} \bar{\sigma}^{k\ell} \frac{\partial}{\partial \bar{z}_\ell}, \ldots \right]$$

SYMMETRIES

- symplectic manifold: any smooth function f defines a Hamiltonian vector field X_f

- X_f preserves the generalized complex structure

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SYMMETRIES

- symplectic manifold: any smooth function f defines a Hamiltonian vector field X_f
- X_f preserves the generalized complex structure
- compact complex manifold: finite dimensional group of symmetries
- however, look at Diff$(M) \ltimes \Omega^2_{exact}(M)$

- complex structure $J = i : [\ldots \partial/\partial z_i \ldots, \ldots d\bar{z}_i \ldots]$

- preserved by $X + \xi \mapsto X + \xi + i_X B \Leftrightarrow B$ type $(1,1)$

- exact + dd^c lemma $\Rightarrow B = dd^c f$

- "Hamiltonian" function f

GENERALIZED KÄHLER MANIFOLDS

Kähler $\Rightarrow$ complex structure + symplectic structure

- complex structure $\Rightarrow J_1$ on $T \oplus T^*$
- symplectic structure $\Rightarrow J_2$ on $T \oplus T^*$
- compatibility ($\omega \in \Omega^{1,1}$): $J_1 J_2 = J_2 J_1$

A **generalized Kähler structure**: two commuting generalized complex structures J_1, J_2 such that $(J_1 J_2(X+\xi), X+\xi)$ is definite.

GUALTIERI'S THEOREM A generalized Kähler manifold is:

- two integrable complex structures I_+, I_- on M
- a metric g, hermitian with respect to I_+, I_-
- a 2-form B
- $U(m)$ connections ∇^+, ∇^- with skew torsion $\pm H = \pm dB$

- $(J_1 J_2)^2 = 1$
- eigenspaces $V, V^\perp$, metric on V positive definite
- connections $\Rightarrow \nabla^+, \nabla^-$ torsion $\pm dB$
- ... also define on $0 \to T^* \to E \to T \to 0$

- $J_1 = -J_2$ on $V \Rightarrow \pm I_+$
- $J_1 = J_2$ on $V^\perp \Rightarrow \pm I_-$
- $\{J_1 = i\} \cap \{J_2 = -i\} = V^{1,0}$ is Courant integrable
- $[X^{1,0} + \xi, Y^{1,0} + \eta] = Z^{1,0} + \zeta \Rightarrow I_+$ integrable

V Apostolov, P Gauduchon, G Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414–428

S. J. Gates, C. M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear σ-models*. Nuclear Phys. B **248** (1984), 157–186.

- $[J_1, J_2] = 0$, $[I_+, I_-] \neq 0$ in general
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- σ is holomorphic

- σ is a holomorphic Poisson structure

EXAMPLES

- T^4 and $K3$, $\sigma =$ holomorphic 2-form (D. Joyce)

- $\mathbf{C}P^2$, $\sigma = \partial/\partial z_1 \wedge \partial/\partial z_2$

- $\mathbf{C}P^1 \times \mathbf{C}P^1 =$ projective quadric.

  $\sigma = \partial Q/\partial z_3[\partial/\partial z_1 \wedge \partial/\partial z_2]$

- moduli space of ASD connections on above.

NJH, *Instantons, Poisson structures and generalized Kaehler geometry*, math.DG/0503432

- Input: holomorphic Poisson structure $\Rightarrow J_1$

- find J_2, symplectic + B-field commuting with J_1

- Output: holomorphic Poisson structures

- Example: input the Hirzebruch surface F_2, output is $F_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$

OPEN QUESTIONS

- When does $[I_+, I_-] = 0$?

- Which complex manifolds admit holomorphic Poisson structures?

  C. Bartocci and E. Macrì, *Classification of Poisson surfaces*, Communications in Contemporary Mathematics, **7** (2005), 1-7

- Which of these admit bihermitian metrics?

- Are I_+, I_- equivalent?

hep-th/0501071 Anton Kapustin, Yi Li, *Open String BRST Cohomology for Generalized Complex Branes*

hep-th/0501062 Roberto Zucchini, *Generalized complex geometry, generalized branes and the Hitchin sigma model*

hep-th/0409250 Ulf Lindstrom, *Generalized complex geometry and supersymmetric non-linear sigma models*

hep-th/0407249 Anton Kapustin, Yi Li, *Topological sigma-models with H-flux and twisted generalized complex manifolds*

hep-th/0405085 U Lindstrom, R Minasian, A Tomasiello, M Zabzine, *Generalized complex manifolds and supersymmetry*