DEFORMATION QUANTIZATIONS

AND

GERBS

JOINT WORK (WORK IN PROGRESS)

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I. WEYL ALGEBRA

W_{2n} : ALGEBRA / $\mathbb{C}$ GENERATED BY

$$z_1, \cdots, z_{2n}$$

with $$[z_i, z_j] = i\hbar J_{ij}$$

WHERE

$$[a, b] := a * b - b * a \quad , \quad a, b \in W_{2n}$$

$*$: THE PRODUCT ON W_{2n}

$$J = (J^{ij}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M(2n; \mathbb{C})$$

Star Exponential Function

$f \in W_{2n}$, CONSIDER

$$\exp_* f := \sum_{k=0}^{\infty} \frac{1}{k!} (\underbrace{f * \cdots * f}_{k}) \quad \text{(formally)}$$

Question

GIVE A MEANING OF STAR EXPONENTIAL FUNCTIONS!

1) FORMAL (W.R.T. $\hbar$)

2) CONVERGENT PROBLEM ($\hbar$: REAL CONST)

LIE ALGEBRA OF QUADRATIC FORMS

$$\mathfrak{Q}^{(2)} = \{ f \in W_{2n} \mid f = \sum_{i,j=1}^{n} a_{ij} \, z_i * z_j ,$$

$$a_{ij} = a_{ji} \in \mathbb{C} \}$$

LEMMA

$(\mathfrak{Q}^{(2)}, \frac{1}{i\hbar}[\ ,\ ])$: complex LIE ALGEBRA

$\cong \ \mathfrak{sp}(n, \mathbb{C})$

QUESTION

CAN CONSTRUCT A LIE GROUP VIA STAR EXPONENTIAL FUNCTIONS ?

$G :=$ generated by $\{ \exp_* f \mid f \in \mathfrak{Q}^{(2)} \}$

$\stackrel{?}{=}$ LIE GROUP

BLURRED LIE GROUP

THEOREM (OMMY)

$\exists\, G$: BLURRED LIE GROUP OF $Sp(n,\mathbb{C})$

1) G : $\mathbb{Z}_2$- GERB OVER $Sp(n,\mathbb{C})$

2) G has PRODUCT STRUCTURE WHICH is smooth

3) G is generated by $\{\exp_* \frac{1}{\hbar} f \mid f \in \mathfrak{a}^{(2)}\}$

NB G is NOT A SMOOTH MANIFOLD

$$
\begin{tikzcd}
\widetilde{Sp(n,\mathbb{R})} = Mp(n,\mathbb{R}) \arrow[r, phantom, "\subset"] & \underline{Mp(n,\mathbb{C})} \simeq \widetilde{Sp(n,\mathbb{C})} \\
Sp(n,\mathbb{R}) \arrow[u, dashed, "\exp_*"] \arrow[r, phantom, "\subset"] & Sp(n,\mathbb{C}) \arrow[u, dashed] \\
\mathfrak{a}^{(2)}_{\mathbb{R}} \cong sp(n,\mathbb{R}) \arrow[u, dash, dotted] \arrow[r, phantom, "\subset"] & sp(n,\mathbb{C}) \cong \mathfrak{a}^{(2)} \arrow[u, dash, dotted]
\end{tikzcd}
$$

QUESTION ANALYZE G GEOMETRICALLY

ORDERING PROBLEMS

<u>LEMMA</u> $W_{2n} \cong \mathcal{P}(\mathbb{C}^{2n})$ (as linear sp.)

<u>Pf.</u> $\forall f \in W_{2n}$

$$\Rightarrow \quad f = \sum a_\alpha z_1^{\alpha_1} * \cdots * z_{2n}^{\alpha_{2n}}$$

(UNIQUELY)

$$a_\alpha \in \mathbb{C}$$

$$\simeq \sum a_\alpha z_1^{\alpha_1} \cdots z_{2n}^{\alpha_{2n}} \in \mathcal{P}(\mathbb{C}^{2n})$$

$*_K$ - product on $\mathcal{P}(\mathbb{C}^{2n})$

$$\Sigma_{2n} = \{ K \in M(2n, \mathbb{C}) \mid {}^tK = K \}$$

For $f, g \in \mathcal{P}(\mathbb{C}^{2n})$, $\in W_{2n}$ $\not\ni$ f

$$f \underset{K}{*} g = \sum \frac{1}{k!} \left(\frac{i\hbar}{2}\right) \sum_{\substack{i_1 \cdots i_k \\ j_1 \cdots j_k}}^{k} \Gamma^{i_1 j_1} \cdots \Gamma^{i_k j_k}$$

$$\times \; \partial_{z_{i_1}} \cdots \partial_{z_{i_k}} f \cdot \partial_{z_{j_1}} \cdots \partial_{z_{j_k}} g$$

where $K \in \Sigma_{2n}$,

$$\Gamma = (\Gamma^{ij}) = (J^{ij} + K^{ij}) \in M(2n, \mathbb{C})$$

LEMMA $(\mathcal{P}(\mathbb{C}^{2n}), *_K)$: ASSOCIATIVE (NONCOMMUTATIVE) ALGEBRA

Ex.

$K = 0 \Longrightarrow$ MOYAL PRODUCT (WEYL ORDERING)

$K = K_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \Longrightarrow$ ΨDOp-product (NORMAL ORDERING)

$K = -K_0 \Longrightarrow$ ANTI-ΨDOp product (ANTI-NORMAL ORDERING)

LEMMA

(1) $(\mathcal{P}(\mathbb{C}^{2n}), *_K) \cong W_{2n}, \quad \forall K \in \Sigma_{2n}$

(2) For $K, K' \in \Sigma_{2n}$,

$\exists\, I_K^{K'} : (\mathcal{P}(\mathbb{C}^{2n}), *_K) \longrightarrow (\mathcal{P}(\mathbb{C}^{2n}), *_{K'})$

s.t. $I_K^{K'}(f *_K g) = I_K^{K'}(f) *_{K'} I_K^{K'}(g)$

(INTER TWINER)

$$I_K^{K'}(f) = \left(\exp \frac{\hbar i}{4} \sum (K^{ij} - K'^{ij})\, \partial_{z_i} \partial_{z_j}\right)(f)$$

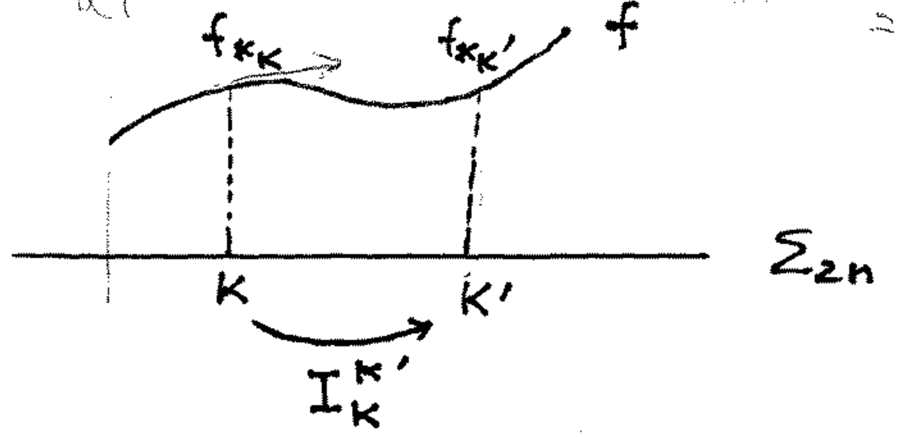
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\hbar i}{4} \sum (K^{ij} - K'^{ij})\, \partial_{z_i} \partial_{z_j}\right)^k (f)$$

INFINITESIMAL INTERTWINER

IDEA WE VIEW $f \in W_{2n}$ AS A SET

$$f = \{ f_{*_K} \}_{K \in \Sigma_{2n}}$$

VIA $(\rho(\mathbb{C}^{2n}), *_K) \cong W_{2n}$



$\left(x, f(x) \right)_{\in W_{2n}}$

$f = \sum a_{ij} z_i *_K z_j$

$= \sum a_{ij} z_i z_j$

$+ \left(\frac{i\hbar}{2} \right) K^{ij} a_{ij}$

f_{*_K}

INF. INTERTWINER

$K \in \Sigma_{2n}$, $\Lambda \in T_K \Sigma_{2n} \cong \Sigma_{2n}$

$dI_K(\Lambda)(f)$

$$:= \frac{d}{dt} I_K^{K+t\Lambda}(f) \Big|_{t=0}$$

$$= \frac{i\hbar}{4} \Big(\sum_{i,j=1}^{n} \Lambda^{ij} \partial_{z_i} \partial_{z_j} \Big)(f)$$

CONNECTION

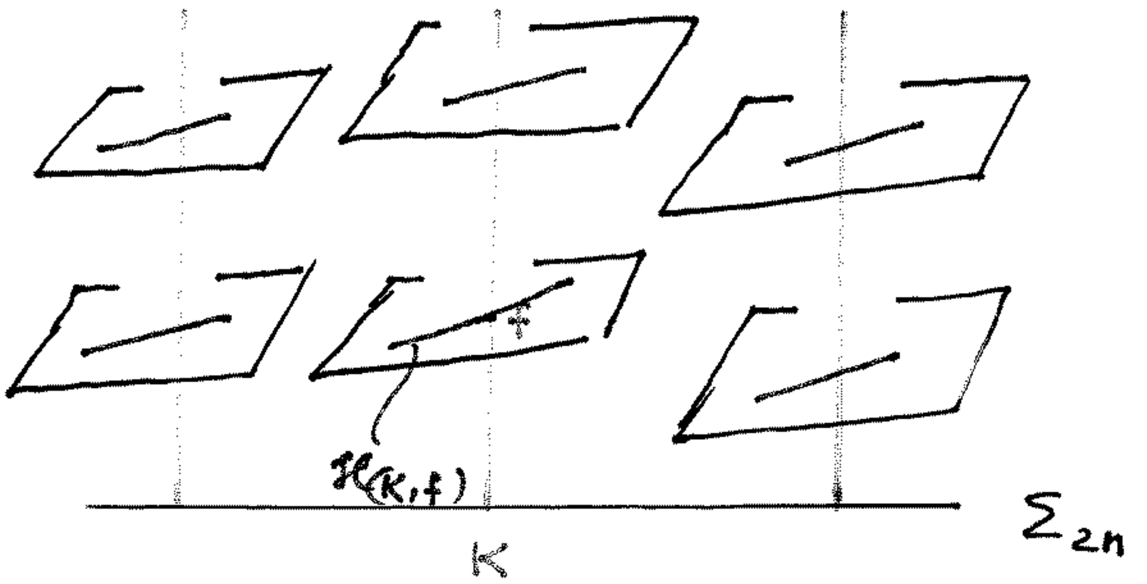
$$L = \coprod_{K \in \Sigma_{2n}} \mathcal{P}_K(\mathbb{C}^{2n}) \quad , \quad \mathcal{P}_K(\mathbb{C}^{2n}) = \mathcal{P}(\mathbb{C}^{2n})$$

$$\begin{array}{c} L \\ \Big\downarrow \pi \\ \Sigma_{2\pi} \end{array}$$

HORIZONTAL SPACE

$K \in \Sigma_{2n}$, $f \in \mathcal{P}(\mathbb{C}^{2n})$

$$\mathcal{H}_{(K,f)} = \{ (\Lambda, dI_K(\Lambda)(f)) \ ; \ \Lambda \in \Sigma_{2n} \}$$



PARALLEL SECTION

$K(t)$: curve in Σ_{2n}

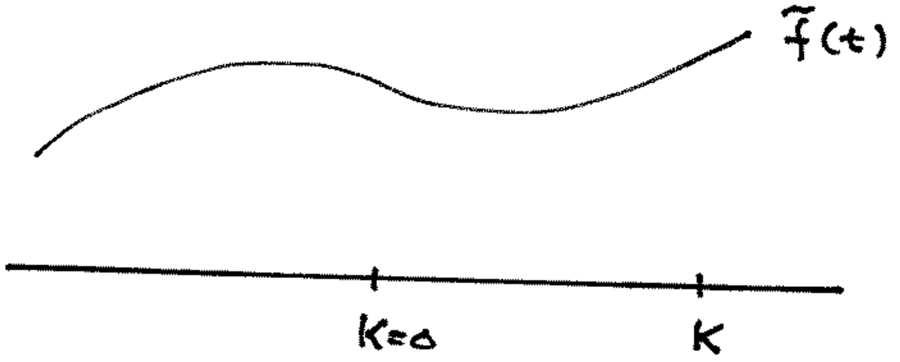
$\tilde{f}(t) \in W_{2n}$: parallel sectión

$\iff$

$$\frac{\partial \tilde{f}}{\partial t} = dI_{K(t)}(\dot{K}(t))(f)$$

e.g.

$\tilde{f}(t) = I_0^{K(t)}(f)$: parallel sectión through $K=0$



QUESTION Extend this NOTION TO ANALYTIC FUNCTIONS / MEROMORPHIC FUNCTIONS, ...

II. STAR EXPONENTIAL FUNCTIONS

$$\mathfrak{a}^{(2)} = \left\{ f \in W_{2n} \;\middle|\; \begin{array}{l} f = \sum a_{ij}\, z_i * z_j \\ a_{ij} = a_{ji} \in \mathbb{C} \end{array} \right\}$$

$f \in \mathfrak{a}^{(2)}$

$$\exp_* tf := F_*(t)$$

Evolution equation

$$(*) \quad \begin{cases} \partial_t F_*(t) = f * F_*(t) \\ F_*(0) = 1 \end{cases}$$

$\Longrightarrow$

$*_K$- rep.

$$(*)_K \quad \begin{cases} \partial_t F_{*_K}(t) = f_{*_K} \mathbin{*_K} F_{*_K}(t) \\ F_{*_K}(0) = 1 \end{cases}$$

$$(1)\quad f_{*_K} = \sum_{i,j=1}^{n} a_{ij} z_i z_j + \frac{i\hbar}{2}\sum_{i,j=1}^{n} K_{ij}\, a_{ij}$$

$$\mathcal{S}^{(2)} = \Big\{ F = \rho \exp Q(z) \;\Big|\; \rho \in \mathbb{C}^{\times},$$
$$Q(z) = \sum_{i,j=1}^{n} Q_{ij} z_i z_j \;,\quad Q_{ij} = Q_{ji} \Big\}$$

<u>Assume</u>

$$(2)\quad F_{*_K}(t) = \rho_K(t) \exp \frac{1}{\hbar} Q_K(t, z) \;\in\; \mathcal{S}^{(2)}$$

PLUGGING (1), (2) into $(*_K)$ WE HAVE AN ANALYTIC SOLUTION OF $(*_K)$

<u>NOTATION</u>

$$A \in \Sigma_{2n} \iff a = A \cdot J \in sp(n, \mathbb{C})$$

$$C_K(a) = (1 + (1+KJ)a)^{-1}(1 - (1-KJ)a)$$

(Twisted Cayley transformation)

$K \in \Sigma_{2n}$

THEOREM

THE SOLUTION OF $(*_K)$ IS GIVEN BY

$$F_{*_K}(t) = \rho_K(t) \exp \tilde{Q}_K(z)$$

where

$$\rho_K \in \mathbb{C}^\times , \quad \tilde{Q}_K(z) = \sum_{i,j=1}^{n} \tilde{Q}_{K,ij}(t)\, z_i z_j$$

$$\tilde{q} = \tilde{Q}\cdot J = C_K^{-1}(e^{-ta}) \quad , \quad a = A\cdot J$$

$$\rho(t) = e^{i\hbar(\sum a_{ij}K_{ij})t} \cdot \left[\det\left((1-KJ) + e^{-ta}(1+KJ) \right) \right]^{-1/2}$$

(contains an ambiguity of choice of sign for $\sqrt{\cdot}$)

Parallel section for $\mathcal{S}^{(2)}$

$K(t)$: curve in Σ_{2n}

$$\begin{cases} \dfrac{\partial}{\partial t} \tilde{Q}(t) = \dfrac{4}{i\hbar} \tilde{Q} \dot{K} \tilde{Q} \\[2ex] \dfrac{\partial}{\partial t} \rho(t) = \left[\dfrac{2}{i\hbar} \mathrm{Tr}(\dot{K}\tilde{Q}) + i\hbar \sum K_{ij}(t)\, a_{ij} \right] \rho(t) \end{cases}$$

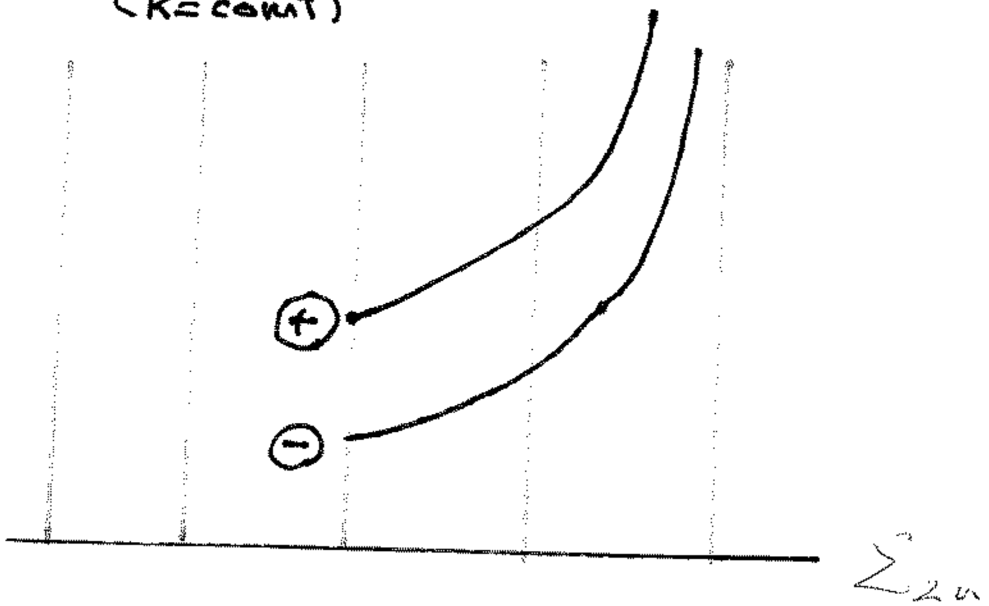
$$(\rho, \tilde{Q}) \in \mathcal{S}^{(2n)} \iff \rho \exp \tilde{Q}(z) \in \mathcal{S}^{(2)}$$

Solution

$$\tilde{Q}(t) = A(1-AK)^{-1}$$

$$\rho(t) = (\det(1-A\dot{K}t))^{-1/2}\,\rho(0)$$: pole, ± ambiguity

$(\dot{K} = \mathrm{const})$



— Multi (±)-valued section

$\dot{K}(t) = \dot{K}\cdot t$

III. TOY MODELS

(1) Phase space of ODEs

simplify PARALLEL Equation!

(A) $\dfrac{dy}{dt} = y^2$ $\qquad (t \in \mathbb{C})$

(B) $\dfrac{dy}{dt} = y^2 , \quad \dfrac{dz}{dt} = -y$ $\quad$ (k y)

Solution space

$$\mathcal{A} = \{ \sigma(t) = (t, y(t)) \mid y(t) : \text{sol'n of (A)} \}$$

$$\mathcal{B} = \{ \tilde{\sigma}(t) = (t, y(t), z(t)) \mid (y(t), z(t)) \text{ of (B)} \}$$

$$\pi_3 : \mathcal{B} \longrightarrow \mathcal{A}$$

$$\tilde{\sigma}(t) = (t, y(t), z(t)) \longmapsto (t, y(t))$$

Question Describe $\pi_3 : \mathcal{B} \longrightarrow \mathcal{A}$

geometrically

<u>Lemma</u> $\mathcal{A} \cong S^2 \;(\mathbb{C} \cup \{\infty\})$

<u>Pf.</u> General solutións for (A)

$$y(t) = \frac{a}{1-at} \qquad (a : \text{const})$$

$$\Updownarrow \quad c = 1/a \quad (a \neq 0)$$

$$y(t) = \frac{1}{c-t} \qquad (c : \text{const})$$

$a = 0$: $(t; 0)$ trivial solution through 0

$c = 0$: $(t, -\frac{1}{t})$ a sotulcón through ∞

The map

$$\tau : \underset{\Psi}{A} \longrightarrow S^2$$

$$\sigma(t) = \left(t, \frac{a}{1-at}\right) \longmapsto a$$

is bijective

ODE (B) : General solutión

$$\tilde{\sigma}(t) = \left(t, \frac{a}{1-at}, z + \underline{\log(1-at)}\right)$$

$$\Updownarrow$$

$$\tilde{\sigma}(t) = \left(t, \frac{1}{c-t}, z + \underline{\log(c-t)}\right)$$

$$(c, z + 2\pi i\mathbb{Z}) \Longleftrightarrow (c^{-1}, z + \pi i + \log c + 2\pi i\mathbb{Z})$$

many-to-many

PROPOSITION

$\pi_3 : \mathcal{B} \longrightarrow \mathbb{C}$ cannot be a fiber bundle

- No Local triviality !
- gluing ambiguity !

INFINITESIMAL GEOMETRY

$\tilde{\sigma} \in \mathfrak{B}$. consider the tangent space at $\tilde{\sigma} = \left(t, \frac{a}{1-at}, z + \log(1-at)\right)$

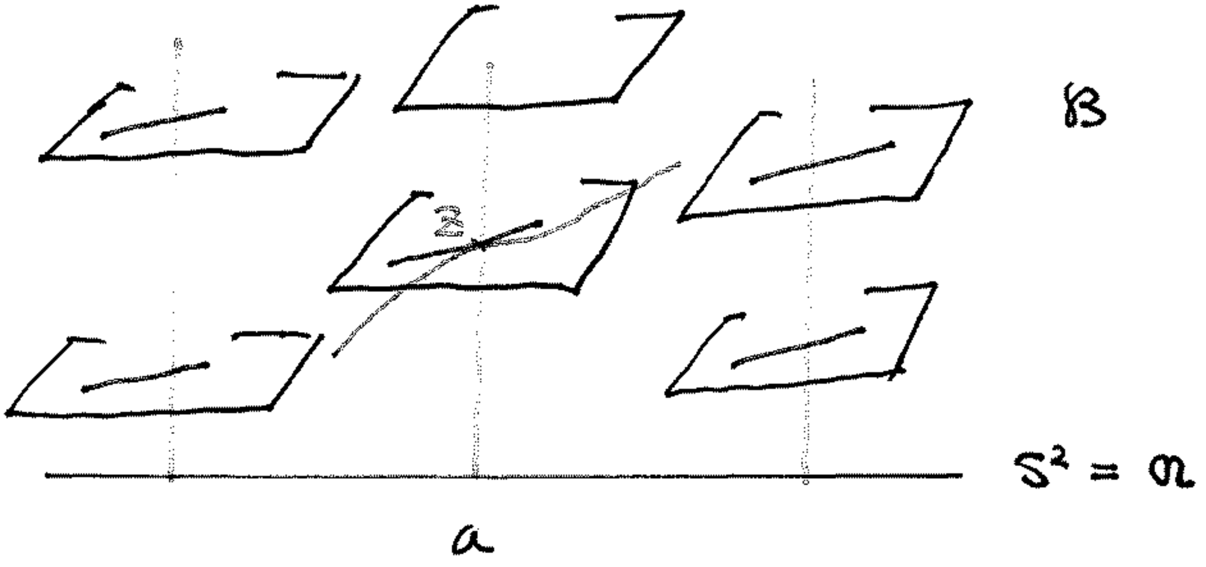
Diff at a:

$$T_{\tilde{\sigma}} \mathfrak{B} = \left\{ X(t) = \left(t, \frac{\alpha}{(1-at)^2}, \beta - \frac{\alpha}{1-at}\right) \;\middle|\; \alpha, \beta \in \mathbb{C} \right\}$$

$$\cup$$

$$\mathcal{H}_{\tilde{\sigma}} = \left\{ X^H(t) = \left(t, \frac{\alpha}{(1-at)^2}, \frac{-\alpha}{1-at}\right) \;\middle|\; \alpha \in \mathbb{C} \right\}$$

Horizontal space



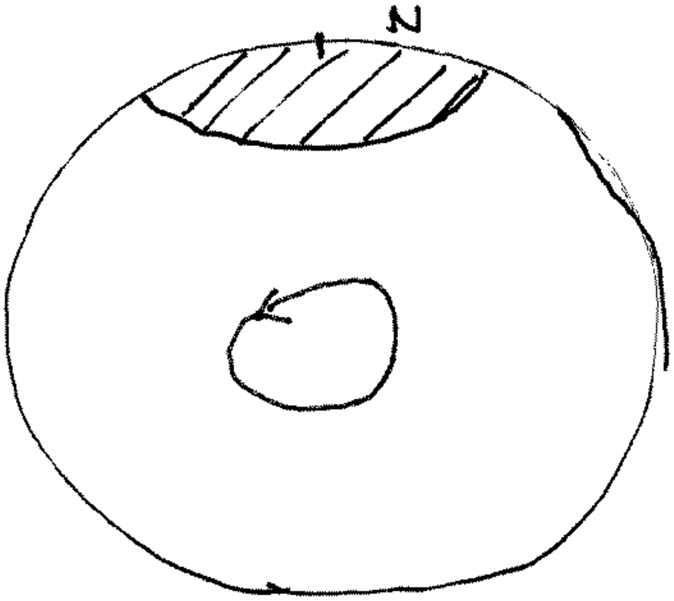
(2) Geometric quantization for non integrable 2-form

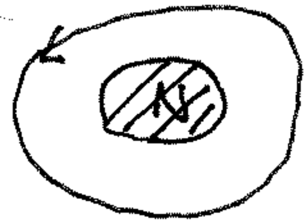
On S^2:

Choose Ω : closed 2-form

s.t $\displaystyle\int_{S^2} \Omega = \mathbb{R}\,(4\pi)$

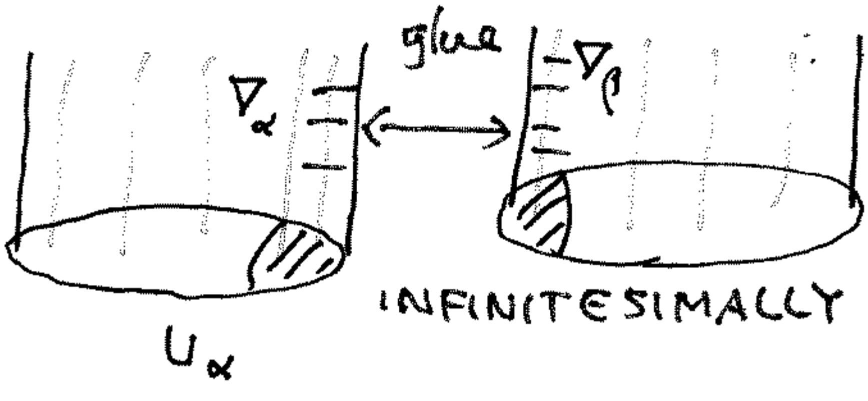
$\operatorname{supp} \Omega \subset V_N$





? $L \longrightarrow S^2$: line bundle with $\frac{1}{2\pi i}\,\mathrm{curv}(\nabla) = \Omega$

Locally



IV. PILES

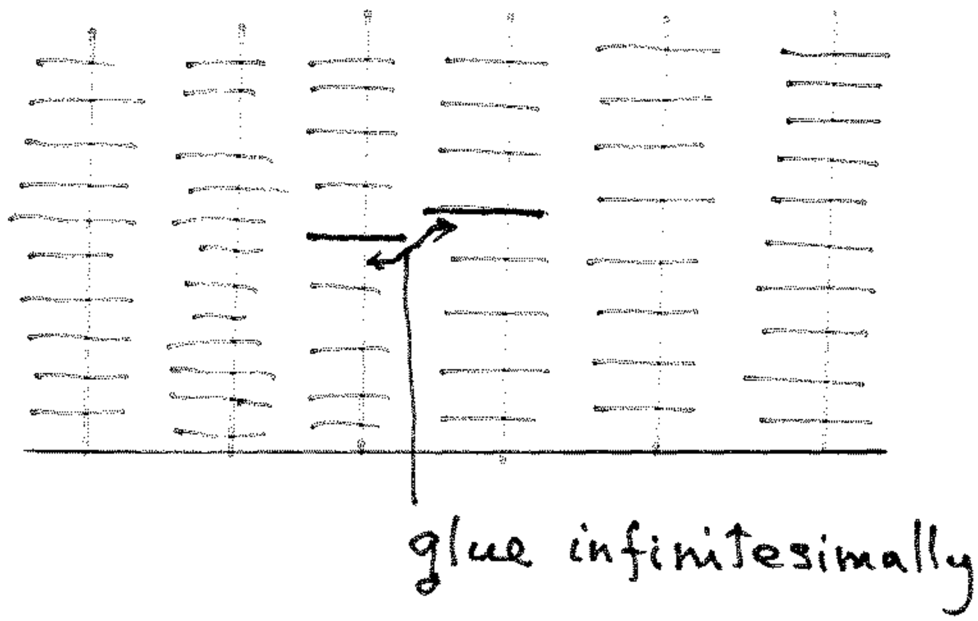
PROBLEM FORMULATE THESE OBJECTS GEOMETRICALLY.

- gerbes

$$\Phi_{\alpha\beta}\,\Phi_{\beta\gamma}\,\Phi_{\gamma\alpha} \neq I$$

- INFINITESIMAL GEOMETRY

VIA CONNECTION



USE: NONLINEAR CONNECTION

REQUIRE :

Multi-valued sectións

1) Phase space of ODE's

Painlévé equation :

characterization of ODE
with no moving branch pts.

⟹

q- Painlévé equation ?

Treat ODE
with moving branch points

[illegible]