BEREZIN-TOEPLITZ QUANTIZATION OF THE MODULI SPACE OF FLAT $SU(n)$ CONNECTIONS

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OUTLINE

1. basics of Berezin-Toeplitz quantization
2. moduli space of flat $SU(n)$ connections in its different guises
3. a certain representation of the mapping class group (MCG, Teichmüller group) on sections of the Verlinde bundle

Results:

1. partly joint with M. Bordemann, E. Meinrenken, and A. Karabegov
2. now classical, a lot of people were involved,
3. report on work of Jørgen Andersen (Aarhus)

BT QUANTIZATION OF COMPACT KÄHLER MANIFOLDS

(M, ω) a (compact) Kähler manifold
i.e. M a complex manifold, ω a Kähler form (closed $(1,1)$ form which is positive)

Examples

1. $\mathbb{C}^n$, $\quad \omega = \mathrm{i}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$
2. $\mathbb{P}^1$, $\quad \omega = \frac{\mathrm{i}}{(1+z\bar{z})^2} dz \wedge d\bar{z}$
3. every Riemann surface
4. every (complex) torus
5. every (quasi-)projective manifold
6. very often moduli spaces

(M, ω) a (compact) Kähler manifold

Quantization condition: (M, ω) is called quantizable, if there exists an associated quantum line bundle (L, h, ∇)

L is a holomorphic line bundle over M,
h a hermitian metric on L,
∇ a compatible connection fullfilling additionally

$$curv_{(L,\nabla)} = -\mathrm{i}\,\omega$$

Note: Not all Kähler manifolds are quantizable
e.g. the tori are only quantizable if they have enough theta functions, i.e. if they are abelian varieties

Consider now $L^m := L^{\otimes m}$, with metric $h^{(m)}$.

$\Gamma_\infty(M, L^m)$ the space of smooth sections
$\Gamma_{hol}(M, L^m) = \mathrm{H}^0(M, L^m)$ the space of global holomorphic sections

scalar product

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\,\Omega, \qquad \Omega := \frac{1}{n!} \underbrace{\omega \wedge \omega \cdots \wedge \omega}_{n}$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

BEREZIN-TOEPLITZ OPERATOR QUANTIZATION

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}(M, L^m)$

$$s \quad \mapsto \quad \Pi^{(m)}(f \cdot s) =: T_f^{(m)}(s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)$$

the Toeplitz operator of level m.

The Berezin-Toeplitz operator quantization is the map

$$f \mapsto \left(T_f^{(m)} \right)_{m \in \mathbb{N}_0} .$$

The BT quantization has the correct semi-classical behavior

SEMI-CLASSICAL BEHAVIOUR

Theorem 1 (Bordemann, Meinrenken, and Schl.)

(a)

$$\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$$

(b)

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

(c)

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| = O(1/m)$$

DEFORMATION QUANTIZATION

Theorem 2 (BMS, Schl., Karabegov and Schl.)

$\exists$ a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f,g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T^{(m)}_{C_k(f,g)}$$

Further properties: it is of separation of variables type,

with classifying Deligne-Fedosov class $\frac{1}{\mathrm{i}}(\frac{1}{\lambda}[\omega] - \frac{\epsilon}{2})$ and Karabegov form $\frac{-1}{\lambda}\omega + \omega_{can}$ □

Existence also implicitely in the work of Boutet-de-Monvel and Guillemin on generalized Toeplitz operators

THE MODULI SPACE OF FLAT $SU(n)$ CONNECTIONS

X an oriented compact surface, $p \in X$ a fixed point,
group $SU(n) = G$

$\mathcal{A}_{F,\xi}$ the set of flat $SU(n)$ connections over $X \setminus \{p\}$ with holonomy ξ around p.
center of $SU(n)$ is $\mathbb{Z}/n\mathbb{Z}$, fix a generator

then ξ corresponds to d mod n

$\mathcal{G}, \qquad X \to G \qquad$ is the gauge group

$$\mathcal{M} := \mathcal{A}_{F,\xi}/\mathcal{G} \cong Hom_d(\tilde{\pi}_1(X), G)/G$$

central extension ($1 \mapsto d$ mod n)

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\pi}_1(X) \longrightarrow \pi_1(X) \longrightarrow 0$$

$\mathcal{M}_S$ is the moduli space of irreducible flat connections, resp. irreducible representations.

Result:

$\mathcal{M}_S$ carries a natural symplectic structure ω, and an associated hermitian line bundle $\mathcal{L}$, which is a quantum line bundle.

It is constructed out of the WZW cocycle of the Chern-Simons action.

So far the symplectic structure

choose a complex structure σ on X:

$X \Longrightarrow X^{\sigma}$ is now a Riemann surface
$\mathcal{M}_s \Longrightarrow \mathcal{M}_s^{\sigma}$, $\omega \Longrightarrow \omega^{\sigma}$ is now a Kähler manifold
$\mathcal{L} \Longrightarrow \mathcal{L}^{\sigma}$ becomes a holomorphic line bundle,
in fact, it is a quantum line bundle.

$\mathcal{M}_s^{\sigma}$ is a quantizable Kähler manifold

What is its geometry?
Compact?

Approach: $X^{\sigma} \quad \Longleftrightarrow \quad$ smooth projective curve over $\mathbb{C}$.

HOLOMORPHIC RANK n BUNDLES E OVER THE SMOOTH PROJECTIVE CURVE C

$\det E := \bigwedge^n E$ determinant line bundle
$\deg(E) := \deg(\det E)$ degree
Does there exists a moduli space of such bundles?
In generally not!
need to consider the subset of (Mumford) stable bundles, resp. S-equivalence classes of semi-stable bundles.

T is a fixed line bundle, $n = \mathrm{rk}\,(E)$,

$U_s(n,d)$, $\deg(E) = d$, E stable
$U_s(n,T)$, $\det(E) = T$, E stable
$U(n,d)$, $\deg(E) = d$, E semi-stable
$U(n,T)$, $\det(E) = T$, E semi-stable

Notation: $d[p]$ is the line bundle corresponding to d times the divisor p

Properties

1. $M := U(n, d[p])$ is always projective algebraic (hence compact)
2. $M_s := U_s(n, d[p])$ is Zariski open and smooth in M, hence a smooth manifold
3. $gcd(n, d) = 1$ implies $M = M_s$, hence a compact Kähler manifold
4. the singularites of M are rather mild
5. $Pic(M_s) = Pic(M) = \mathbb{Z} \cdot [L]$, where L is a special ample line bundle
6. $\Gamma_{hol}(M_s, L^m_{|}) = \Gamma_{hol}(M, L^m)$
7. $g = 2$ and $n = 2$ $\quad M$ is always smooth

$$\boxed{\mathcal{M}_s^\sigma \cong U_s(n, d[p]) = M_s}$$

as complex manifold and as Kähler manifolds,

$$\boxed{\mathcal{L}^\sigma \cong L}$$

as holomorphic line bundles.
Names: Narasimhan, Seshadri, Weil, Mumford, ...

$\mathrm{H}^0(M, L^m) = \Gamma_{hol}(M, L^m)$ are called Verlinde spaces and the dimension formula (as function of m) is called the Verlinde formula.

these vector spaces are the quantum spaces
we have the BT quantum operators

$$T_f^{(m)} : \mathrm{H}^0(M, L^m) \to \mathrm{H}^0(M, L^m)$$

we can apply the BMS theorem und use the natural deformation quantization $\star_{BT}$ (at least if $M = M_s$, resp. M smooth)

We have to go one step further

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\text{choose } \sigma"] & (\mathcal{M}_s, \mathcal{L}^m) \arrow[r] \arrow[d] & \Gamma_\infty(\mathcal{M}_s, \mathcal{L}^m) \arrow[d, "\Pi^{\sigma,(m)}"] \\
X^\sigma \arrow[r] & (\mathcal{M}_s^\sigma = M_s, (\mathcal{L}^\sigma)^m = L^m) \arrow[r] & \Gamma_{hol}(M_s, L^m)
\end{tikzcd}$$

$\mathcal{T}$ Teichmüller space, (space of all complex structures on X modulo a certain equivalence relation)
over $\mathcal{T}$ there is the trivial (infinite dimensional) bundle with fibre $\Gamma_\infty(\mathcal{M}, \mathcal{L}^m)$

contains the subbundle $\mathcal{V}_m$ with fibre $\Gamma_{hol}(M_s, L^m)$ over $\sigma \in \mathcal{T}$.

$\mathcal{V}_m$ is called the Verlinde bundle over $\mathcal{T}$

Given $f \in C^\infty(\mathcal{M}_s)$

$$\left(T^{(m)}_{f,\sigma}\right)_{\sigma\in\mathcal{T}}$$

is a family of operators on the Verlinde bundle,

i.e. $T^{(m)}_{f,\,.}$ is a section of $End(\mathcal{V}_m)$

THE MAPPING CLASS GROUP ACTION

$\mathcal{T}$ Teichmüller space

$\mathcal{V}_m$ and $End(\mathcal{V}_m)$ bundles over $\mathcal{T}$.

1. there exists a certain projectively flat connection ∇ on $\mathcal{V}_m$ Axelrod, della Pietra, Witten – Hitchin connection,
2. MCG operates on the covariantly constant sections of $\mathbb{P}(\mathcal{V}_m)$,
3. J. Andersen showed this action of the MCG is asymptotically faithful (i.e. given an element of MCG, there is an *m* such that its action is non-trivial)

Definition of the mapping class group(MCG)

$$\Gamma := \mathit{MCG} := \mathit{Diff}^+(X)/\mathit{Diff}_0(X)$$

here X is the surface of genus g,
$\mathit{Diff}^+(X)$ orientation preserving diffeomorphisms
$\mathit{Diff}_0(X)$ diffeomorphisms isotop to the identity

- per definition Γ operates on the surface X
- it operates on the Teichmüller space, in fact the moduli space $\mathcal{M}_g$ of genus g curves is the quotient $\mathcal{T}/\Gamma$.
- it operates on the fundamental group $\pi_1(X)$, and on $\mathit{Hom}_d(\tilde{\pi}_1(X), G)/G$
- and furthermore on $\mathcal{M}_s^\sigma \cong M_s$, the moduli spaces of irreducible connections, resp. stable bundles

∇ the AdPW-H connection on $\mathcal{V}_m$ (projectively flat)
$\mathbb{P}(W_m)$ space of covariantly constant sections of $\mathbb{P}(\mathcal{V}_m)$
∇^{end} induces a flat connection on $End(\mathcal{V}_m)$
Γ operates on $\mathbb{P}(W_m)$: $\rho_m : \Gamma \to Aut(\mathbb{P}(W_m))$

Theorem (Andersen)
For $g \geq 3$ the map ρ_m is asymptotically faithful

More precisely,

$$\bigcap_{m=1}^{\infty} ker(\rho_m) = \begin{cases} 1, & g > 2, \text{ or } g = 2, n > 2, \text{ or} \\ & g = 2, n = 2, d \text{ odd} \\ \{1, H\}, & g = 2, n = 2, d \text{ even} \end{cases}$$

H is the hyperelliptic involution

IMPORTANCE

The assignment

$$X \quad \longrightarrow \quad V(X) = \mathrm{H}^0(M_s, L^m)$$

corresponds to a TQFT

should be independent of the complex structure chosen

the projectively flat connection gives locally a natural identification

globally the choice reduces to on action of the mapping class group Γ – (also a topological invariant)

gives invariants of the TQFT

RELATION TO BT

Note: $f \in C^\infty(\mathcal{M}_s)$, i.e. a smooth function on the moduli space of connections

Proposition A (Andersen)

$\sigma_0, \sigma_1 \in \mathcal{T}$,

$P^{end}_{\sigma_0,\sigma_1}$ parallel transport from σ_0 to σ_1 in $End(\mathcal{V}_m)$ then

$$||P^{end}_{\sigma_0,\sigma_1} T^{(m)}_{f,\sigma_0} - T^{(m)}_{f,\sigma_1}|| = O(1/m)$$

(uses BMS, Def.quantization and further hard work)

Proposition B (Andersen)

$\phi \in \Gamma$, $\phi \in ker\, \rho_m$ then

$$T^{(m)}_{f,\sigma} = P^{end}_{\phi(\sigma),\sigma} T^{(m)}_{f\circ\phi,\phi(\sigma)}$$

Theorem (Andersen)
$\phi \in \Gamma$, $\phi \in \bigcap_m \ker \rho_m$ then ϕ induces the identity on $\mathcal{M}_s$.

Some indication of the proof:
By Proposition B

$$T^{(m)}_{f - f\circ\phi,\sigma} = T^{(m)}_{f,\sigma} - T^{(m)}_{f\circ\phi,\sigma} = P^{end}_{\phi(\sigma),\sigma} T^{(m)}_{f\circ\phi,\phi(\sigma)} - T^{(m)}_{f\circ\phi,\sigma}$$

Take the norm of this expression and use Prop. A

$$||T^{(m)}_{f - f\circ\phi,\sigma}|| = O(1/m)$$

Or,

$$\lim_{m\to\infty} ||T^{(m)}_{f - f\circ\phi,\sigma}|| = 0$$

what implies $|f - f\circ\phi|_\infty = 0$ by BMS, part a for all f,
hence $\phi = id$. □

SOME LITERATURE

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APPENDIX: SYMPLECTIC FORM ON $\mathcal{M}$

$\mathcal{A}_F$ the affine space of all flat $SU(n)$ connections
$\mathfrak{g} = su(n)$, $\alpha, \beta \in \Omega^1(X) \otimes \mathfrak{g}$ are tangent vectors

$$\Omega_A(\alpha, \beta) = \frac{\mathrm{i}}{2\pi} \int_X \mathrm{Tr}(\alpha \wedge \beta)$$

is a skew-symmetric form on $\mathcal{A}_F$ ($A \in \mathcal{A}_F$)
descends to $\mathcal{A}_F / \mathcal{G}$
restricted to $\mathcal{M}_s = \mathcal{A}_F^s / \mathcal{G}$ it is a symplectic form

bundle:

One uses the CS action:

N a 3-manifold with $\partial N = X$.

$\tilde{A}$ any connection on N

$$CS(\tilde{A}) = \frac{1}{4\pi}\int_N \mathrm{Tr}(\tilde{A}\wedge d\tilde{A} + \frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A})$$

$A \in \mathcal{A} \quad \longrightarrow \tilde{A}$ an extension to N

$g \in \mathcal{G} \quad \longrightarrow \tilde{g}$ an extension to $N \to SU(n)$

$\theta(A, g) := \exp(\mathrm{i}(CS(\tilde{A}^{\tilde{g}}) - CS(\tilde{A}))$ is a $U(1)$-valued welldefined cocycle

$\mathcal{L} := (\mathcal{A}_F^s \times \mathbb{C})/\sim \quad \to \quad \mathcal{A}_F^s/\mathcal{G} = \mathcal{M}_s$
where $(A, z) \sim (A^g, \theta(A, g)z)$
defines a line bundle over $\mathcal{M}_s$

$\eta(\alpha) = \frac{1}{4\pi} \int_X \mathrm{Tr}(A \wedge \alpha)$ induces a connection on $\mathcal{L}$.