Dirac structures

and

the phase space of particles in a Yang-Mills-Higgs field

by

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1. Introduction

• Let G be a Lie group with Lie algebra $\mathfrak{g}$,

• $Q \xrightarrow{\pi} M$ a principal G-bundle,

• $(F, \mathcal{V}_F)$ a Poisson manifold with a Hamiltonian G-action and momentum map

$$J_F : F \to \mathfrak{g}^*.$$

• Let $P \subset T^*M \times Q$ be the pull-back of Q by the canonical projection $T^*M \to M$, i.e. the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{pr_2} & Q \\ pr_1 \downarrow & & \downarrow \pi \\ T^*M & \longrightarrow & M \end{array}$$

Fact: *Every connection on Q induces a Poisson structure on the associated bundle $P \times_G F$.*

Remark.

- When $(F, \mathcal{V}_F)$ is symplectic, the associated bundle $P \times_G F$ is Sternberg's classical phase space for a particle in a Yang-Mills field.

- When $(F, \mathcal{V}_F)$ is a Lie-Poisson manifold, one gets a *gauged Poisson structure* on $P \times_G F$. This case was considered by Marsden, Montgomery and Ratiu.

Summarizing, the pull-back bundle P is a principal G-bundle over the symplectic manifold $(T^*M, \omega_{\text{can}})$, moreover every connection on Q induces a Poisson structure on $P \times_G F$.

Motivated by the fact that pre-symplectic structures naturally arise in the study of the Hamiltonian dynamics of particles with gauge degrees of freedom, we would like to understand the geometric object induced on $P \times_G F$ if one replaces $P \to T^*M$ by a principal G-bundle whose base manifold is pre-symplectic.

A triple (G, P, F) formed by

– a Lie group G,

– a principal G-bundle $P \xrightarrow{\pi} B$,

– and a Hamiltonian Poisson G-space F,

is called a **classical Yang-Mills-Higgs setup.**

__**Theorem 1.**__ *Let (G, P, F) be a classical Yang-Mills-Higgs setup. Assume that the base manifold $B = P/G$ is equipped with a pre-symplectic form ω_B. Then every connection θ on P induces a coupling Dirac structure on the associated bundle $E = P \times_G F$ which restricts to the Poisson structure along the fibers of E inherited from the Poisson manifold $(F, \mathcal{V}_F)$.*

2. Coupling Dirac structures

Let N be a smooth manifold. Consider the symmetric pairing $\langle\cdot,\cdot\rangle$ on the vector bundle $TN \oplus T^*N$ defined by:

$$\langle (X_1,\alpha_1),(X_2,\alpha_2)\rangle = \frac{1}{2}\Big(\alpha_1(X_2)+\alpha_2(X_1)\Big)$$

We also consider the *Courant bracket* on the space of smooth sections of $TN \oplus T^*N$, i.e.

$$[(X_1,\alpha_1),\ (X_2,\alpha_2)] = ([X_1,X_2],\ \mathcal{L}_{X_1}\alpha_2 - i_{X_2}d\alpha_1),$$

for all $(X_1,\alpha_1),\ (X_2,\alpha_2)$ smooth sections of $TN \oplus T^*N$.

A **Dirac structure** on N is a sub-bundle $L \subset TN \oplus T^*N$ which is maximally isotropic with respect to the symmetric pairing $\langle\cdot,\cdot\rangle$ and whose space of sections is closed under the Courant bracket.

Geometric data on a fiber bundle $\pi : E \to B$ consist of a triple $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ formed by an Ehresmann connection Γ, a vertical bivector field $\mathcal{V}$, and a horizontal 2-form $\overline{\mathbb{F}}$ on E.

We say that $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ is **integrable** if the following properties are satisfied:

- $\mathcal{V}$ is a Poisson tensor, i.e. $[\mathcal{V}, \mathcal{V}] = 0$;
- $\mathcal{V}$ is preserved by parallel transport, i.e. $\mathcal{L}_{hor_\Gamma(X)}\, \mathcal{V} = 0,\ \forall X \in \mathfrak{X}(B)$;
- For any $X, Y \in \mathfrak{X}(B)$, $\mathrm{Curv}_\Gamma(X, Y)$ is a Hamiltonian vector field with respect to $(E, \mathcal{V})$ given by

$$\mathrm{Curv}_\Gamma(X, Y) = \mathcal{V}^\sharp(d(\overline{\mathbb{F}}(hor_\Gamma(X), hor_\Gamma(Y)));$$

- $\overline{\mathbb{F}}$ is horizontally-closed.

Let $\pi : E \to B$ be a smooth fiber bundle. A Dirac structure L on E is called a **coupling Dirac structure** if there exist geometric data $(\mathbb{F}, \Gamma, \mathcal{V})$ such that

$$L = \Big\{ (\overline{X}, i_{\overline{X}}\overline{\mathbb{F}}) + (\mathcal{V}^\sharp \alpha, \alpha) \mid \overline{X} \in \mathrm{Hor}_\Gamma,\ \alpha \in \mathrm{Ann}(\mathrm{Hor}_\Gamma) \Big\}.$$

This is a special case of the notion of a coupling Dirac structure on a foliated manifold, recently introduced by Vaisman. Coupling Dirac structures appeared, naturally, when we studied the local structure of Dirac manifolds.

Proposition. *Let $\pi : E \to B$ be a smooth fiber bundle. The integrability of any given geometric data $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ is equivalent to the fact that the space of smooth sections of the corresponding sub-bundle $L \subset TE \oplus T^*E$ (defined as above) is closed under the Courant bracket.*

3. Proof of Theorem 1:

Under the notations and assumptions of Theorem 1, the connection θ on P induces a connection Γ on $E = P \times_G F$. We have the splitting

$$TE = \mathrm{Hor}_\Gamma \oplus \mathrm{Vert}.$$

Moreover, the Γ-horizontal lift of $X \in \mathfrak{X}(B)$ is given by

$$hor_\Gamma(X)_{[p,f]} = T_{(p,f)}\pi_{_{P\times F}}(\overline{X}_p, \mathbf{0}_f), \tag{1}$$

where $\pi_{_{P\times F}} : P \times F \to P \times_G F$ is the canonical projection, $\overline{X}$ is the θ-horizontal lift of $X \in \mathfrak{X}(B)$, and $\mathbf{0}_f$ is the zero tangent vector at f. Define the vertical bivector field $\mathcal{V}$ as follows

$$\mathcal{V} = (\pi_{_{P\times M}})_* \mathcal{V}_F \tag{2}$$

We have the following lemma:

Lemma 1 Under the above notations, we have:

- $\mathcal{V}$ is a vertical Poisson tensor;
- $\mathcal{L}_{hor_\Gamma(X)}\, \mathcal{V} = 0$, for any $X \in \mathfrak{X}(B)$.

Proof: The fact that the Schouten bracket $[\mathcal{V}, \mathcal{V}]$ vanishes follows immediately from

$$[\mathcal{V}_F, \mathcal{V}_F] = 0.$$

Furthermore, it follows from Equations (1)-(2) that one has $\mathcal{L}_{hor_\Gamma(X)}\, \mathcal{V} = 0,$ for any $X \in \mathfrak{X}(B)$.

$\square$

Remark: In the definition of geometric data $(\mathcal{V}, \Gamma, \overline{\mathbb{F}})$ on $E \to B$, we may replace the horizontal 2-form $\overline{\mathbb{F}}$ by the element $\mathbb{F} \in \Omega^2(B) \otimes C^\infty(E)$ defined by

$$\mathbb{F}(X,Y) = \overline{\mathbb{F}}\Big(hor_\Gamma(X), hor_\Gamma(Y)\Big).$$

Define the operator
$\partial_\Gamma : \Omega^k(B) \otimes C^\infty(E) \to \Omega^{k+1}(B) \otimes C^\infty(E)$ as follows

$$\partial_\Gamma \mathbb{G}(X_0, ..., X_k) = \sum_{i=0}^{k} (-1)^i \mathcal{L}_{hor_\Gamma(X)}(\mathbb{G}(X_0, ..., \widehat{X_i}, ..., X_k))$$
$$+ \sum_{i<j} (-1)^{i+j} \mathbb{G}([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k).$$

The fact that $\overline{\mathbb{F}}$ is horizontally closed can be alternatively expressed by the following equation

$$\partial_\Gamma \mathbb{F} = 0.$$

$\square$

End of the proof of Theorem 1:

Taking into account Lemma 1, it only remains to find a suitable 2-form $\overline{\mathbb{F}}$. To construct such a 2-form, we will use the momentum map associated with the Hamiltonian G-action on F, denoted by $J : F \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. Recall that the infinitesimal action $\varrho_F : \mathfrak{g} \to \mathfrak{X}(F)$ transforms an element $\xi \in \mathfrak{g}$ into a Hamiltonian vector field $\mathcal{V}_F^\sharp(dJ_\xi)$, where

$$J_\xi(f) = \langle J(f),\ \xi \rangle,$$

for all $m \in M$. Consider the connection 1-form $\theta \in \Omega^1_{\mathrm{Vert}}(P) \otimes \mathfrak{g}$. The curvature of θ is the horizontal $\mathfrak{g}$-valued 2-form denoted by Curv_θ.

Define

$$\Big(\mathbb{G}(X,Y)\Big)([p,f]) = \Big\langle J(f),\ \mathrm{Curv}_\theta\Big(\overline{X}_p,\ \overline{Y}_p\Big)\Big\rangle, \quad (3)$$

for all $X, Y \in \mathfrak{X}(B)$.

Let ω_B be the pre-symplectic form on B. Define

$$\mathbb{F} = \omega_B \otimes 1 + \mathbb{G}. \quad (4)$$

We will show that $\mathrm{Curv}_\Gamma(X,Y)) = \mathcal{V}^\sharp\Big(d(\mathbb{F}(X,Y))\Big)$, for all $X, Y \in \mathfrak{X}(B)$.

Moreover, the curvature of θ and that of Γ are related as follows

$$\Big(\mathrm{Curv}_{\Gamma}(X,Y)\Big)([p,f]) = \tag{5}$$

$$T_{(p,f)}\pi_{_{P\times F}}\Big(\mathbf{0}_p,\ (\varrho_F \circ \mathrm{Curv}_{\theta}(\overline{X}_p,\overline{Y}_p))(f)\Big),$$

where $\mathbf{0}_p$ is the zero tangent vector at p, ϱ_F is the infinitesimal action associated to the G-action on F. Since it is enough to work with local coordinates, we pick a local system of Darboux-coordinates $(x_1,\ldots,x_{2s})$ on B, i.e $\omega_B = \sum dx_i \wedge dx_{s+i}$. Set

$$\mathbb{G}_{ij} = \mathbb{G}\Big(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\Big) \quad \text{and} \quad \mathbb{F}_{ij} = \mathbb{F}\Big(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\Big).$$

From Equations (3) and (5), one gets

$$\mathrm{Curv}_{\Gamma}\Big(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\Big)([p,f]) \quad = \quad \mathcal{V}^{\sharp}(d\mathbb{F}_{ij})([p,f]).$$

Moreover,

$$\partial_\Gamma \mathbb{F} = d\omega_B \otimes 1 + \partial_\Gamma \mathbb{G}.$$

Since ω_B is closed, we get $\partial_\Gamma \mathbb{F} = \partial_\Gamma \mathbb{G}$. To show that $\partial_\Gamma \mathbb{G} = 0$, we use classical arguments. Precisely, we define the vertical 1-form Φ on $P \times_G F$ as follows:

$$\Phi_{[p,f]}([(X_p, Z_f)]) = \Big\langle J(f),\ \theta_p(X_p) \Big\rangle, \tag{6}$$

where $[(X_p, Z_f)] = T_{(p,f)}\pi_{_{P\times F}}(X_p, Z_f)$. One can easily check that

$$\mathbb{G}(X, Y) = d\Phi(hor_\Gamma(X), hor_\Gamma(Y)).$$

Hence $\partial_\Gamma \mathbb{G} = 0$ since $d^2\Phi = 0$. Consequently, the triple $(\mathcal{V}, \Gamma, \mathbb{F})$ defines integrable geometric data on $E = P \times_G F$.

□

Corollary. (Weak coupling Poisson structures)

Let (G,P,F) be a classical Yang-Mills-Higgs setup and θ a connection on $P \to B$. Assume that both F and P are compact and the base B is equipped with a symplectic form ω_B. Then there is 1-parameter family of Poisson structures Π_ε on the associated bundle $E = P \times_G F$ such that each Poisson structure Π_ε restricts to the Poisson structure along the fibers of E which is inherited from $(F, \mathcal{V}_F)$.

Proof: The additional condition saying that P and F are compact ensures that we can choose a real number $\varepsilon > 0$ sufficiently small so that the 2-form

$$\mathbb{F}_\varepsilon = \omega_B \otimes 1 + \varepsilon \mathbb{G}$$

is non-degenerate. Replacing $\mathbb{F}$ by $\mathbb{F}_\varepsilon$ in the proof of Theorem 1, we conclude that the triple $(\mathcal{V}, \Gamma, \mathbb{F}_\varepsilon)$ is integrable. Its corresponding coupling Dirac structure L_ε satisfies $L_\varepsilon \cap (TE \oplus \{0\}) = \{0\}$. This is equivalent to the fact that L_ε is the graph of a Poisson structure Π_ε. There follows our Corollary. $\square$

4. Functorial property

It is known that the construction of the phase space of a classical particle in a Yang-Mills field is functorial with respect to morphisms of principal bundles and Poisson manifolds. We will show that this functorial property extends to coupling Dirac structures.

Let $\lambda = (G, P, F)$ and $\lambda' = (G', P', F')$ be two classical Yang-Mills-Higgs setups. Assume that $B = P/G$ and $B' = P'/G'$ are equipped with their pre-symplectic form $\omega_{_B}$ and $\omega_{_{B'}}$, respectively. Consider a connection θ (resp. θ') on P (resp. P').

A **morphism from** $(\lambda, \omega_B, \theta)$ **to** $(\lambda', \omega_{B'}, \theta')$ is a triple (ϕ, f, h), where

- $\phi : G \to G'$ is a homomorphism of Lie groups,
- $f : F \to F'$ is a ϕ-equivariant Poisson map
- and $h : P \to P'$ is a ϕ-equivariant map which descends to a morphism $\underline{h} : B \to B'$ of pre-symplectic manifolds satisfying

$$hor_{\theta'}(\underline{h}_* X) = h_*(hor_\theta(X)), \quad \forall X \in \mathfrak{X}(B)$$

and such that the following diagram commutes

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ J\downarrow & & \downarrow J' \\ \mathfrak{g}^* & \xleftarrow{\phi^*} & \mathfrak{g}'^* \end{array}$$

where $\phi^* : \mathfrak{g}'^* \to \mathfrak{g}$ is the canonical map induced by ϕ.

Given such two classical Yang-Mills-Higgs setups, we denote by $L(\lambda, \omega_{_B}, \theta)$ and $L(\lambda', \omega_{_{B'}}, \theta')$ the corresponding Dirac structures on $P \times_G F$ and $P' \times_{G'} F'$, respectively.

Proposition 1 *Every morphism* (ϕ, f, h) *from* $(\lambda, \omega_{_B}, \theta)$ *to* $(\lambda', \omega_{_{B'}}, \theta')$ *induces a Dirac map from* $(P \times_G F,\ L(\lambda, \omega_{_B}, \theta))$ *to* $(P' \times_{G'} F',\ L(\lambda', \omega_{_{B'}}, \theta'))$*.*

Recall that if (N_1, L_1) and (N_2, L_2) are two Dirac manifolds then a smooth map ψ from (N_1, L_1) to (N_2, L_2) is called a **forward Dirac map** if

$$L_2 = \Big\{ (T\psi(X), \beta) | \quad X \in TN_1,\ \beta \in T^*N_2, \\ \text{and } (X, (T\psi)^*\beta) \in L_1 \Big\}.$$

It is called a **backward Dirac map** if

$$L_1 = \Big\{ (X, (T\psi)^*\beta) | \quad X \in TN_1,\ \beta \in T^*N_2, \\ \text{and } (T\psi(X), \beta) \in L_2 \Big\}.$$

A *Dirac map* is a composite of a finite number of forward and backward Dirac maps.

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